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Valuated Matroids

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Oriented matroids have been introduced in [R. G. Bland and M. Las Vergnas, Orientability of Matroids, *J. Combin. Theory Ser. B* **24** (1978), 94–123]. They can be viewed as an abstraction of matroids representable over an ordered field. Analogously, we define valuated matroids as an abstraction of matroids which are representable over some field having a non-archimedian valuation. We study projective equivalence of valuations of matroids and show that valuated matroids correspond in a one-to-one fashion to matroids with coefficients in certain coefficient domains. © 1992 Academic Press, Inc.

INTRODUCTION

It is well known that *orderings* and non-archimedian (or "p-adic") *valuations* of fields have to be viewed as closely related concepts. We know this from algebraic number theory where many results become transparent only if orderings and valuations are treated completely in parallel. It is also true that in convex geometry, that is, geometry over *ordered* fields, many basic results like, e.g., the Farkas Lemma can be reformulated to yield similarly basic results for the corresponding geometry over *valuated* fields. In consequence, parallel to the theory of oriented matroids, that is, the systematic study of combinatorial properties of signs of $m \times m$-subdeterminants of some fixed $m \times n$-matrix over an ordered field, there should exist a corresponding theory of combinatorial properties of the (p-adic) values of such subdeterminants in case one works over a valuated field. It is the purpose of this note to develop such a theory.

As in oriented matroid theory the combinatorial properties in question can be derived quite easily from the famous Grassmann–Plücker identities which state that for any field K and any family $e_1, ..., e_m, f_1, ..., f_m \in K^m$ of m-vectors over K the product $\det(e_1, ..., e_m) \cdot \det(f_1, ..., f_m)$ coincides with the sum

$$\sum_{i=1}^{m} \det(e_1, ..., e_{i-1}, f_1, e_{i+1}, ..., e_m) \cdot \det(e_i, f_2, ..., f_m).$$

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If K is ordered, this implies that whenever $\det(e_1, ..., e_m) \cdot \det(f_1, ..., f_m)$ is positive there must exist some $i \in \{1, ..., m\}$ such that the corresponding summand $\det(e_1, ..., e_{i-1}, f_1, e_{i+1}, ..., e_m) \cdot \det(e_i, f_2, ..., f_m)$ is also positive.

Similarly, assume Γ to be a (multiplicatively written) linearly ordered abelian group and put $\bar{\Gamma} := \{0\} \cup \Gamma$ with $0 \leqslant \gamma$ and $0 \cdot \gamma = 0$ for every $\gamma \in \bar{\Gamma}$. If $\Phi: K \to \bar{\Gamma}$ is a non-archimedian valuation of K, that is, if for all $x, y \in K$ one has

$$\Phi(x) = 0 \Leftrightarrow x = 0,$$

$$\Phi(x \cdot y) = \Phi(x) \cdot \Phi(y),$$

and

$$\Phi(x + y) \leqslant \max(\Phi(x), \Phi(y)),$$

and therefore

$$\Phi(x) = \Phi(-x),$$

as well as

$$\Phi(x + y) = \Phi(x), \qquad \text{if} \quad \Phi(y) < \Phi(x),$$

then for any $e_1, ..., e_m, f_1, ..., f_m \in K^m$ as above there must exist some $i \in \{1, ..., m\}$ with

$$\Phi(\det(e_1, ..., e_m)) \cdot \Phi(\det(f_1, ..., f_m))$$
$$\leqslant \Phi(\det(e_1, ..., e_{i-1}, f_1, e_{i+1}, ..., e_m)) \cdot \Phi(\det(e_i, f_2, ..., f_m)).$$

Consequently, we will define a valuated matroid M of rank m with values in Γ to consist of a pair $M = (E, v)$ with the following properties: E is a—possibly infinite—set on which M is defined and $v: \binom{E}{m} \to \bar{\Gamma}$ is a map from the set $\binom{E}{m} := \{B \subseteq E \mid \# B = m\}$ of subsets B of E of cardinality m into $\bar{\Gamma} = \{0\} \cup \Gamma$ such that $v(B) \neq 0$ for at least some $B \in \binom{E}{m}$ and such that for all $B, B' \in \binom{E}{m}$ and every $f \in B' \backslash B$ there exists some $e \in B \backslash B'$ with

$$v(B) \cdot v(B') \leqslant v(\{f\} \cup (B \backslash \{e\})) \cdot v(\{e\} \cup (B' \backslash \{f\})).$$

If we replace the multiplicative group Γ by the additive group $\mathbb{R}$ and, hence, $\bar{\Gamma}$ by $\{-\infty\} \cup \mathbb{R}$, such maps have been considered already in [DW4] in the context of greedy algorithms. More precisely, we consider for any finite set E and any map $v: \binom{E}{m} \to \{-\infty\} \cup \mathbb{R}$ with $v(B) \neq -\infty$ for at least some $B \in \binom{E}{m}$ the obvious greedy algorithm which, starting with some $B=\{e_1, ..., e_m\} \in \binom{E}{m}$ with $v(B) \neq -\infty$ replaces the $e_1, ..., e_m$ consecutively by some $f_1, ..., f_m \in E$ with

$$v(\{f_1, ..., f_{i-1}, f_i, e_{i+1}, ..., e_m\}) \geqslant v(\{f_1, ..., f_{i-1}, f, e_{i+1}, ..., e_m\})$$

for all $f \in E \backslash \{f_1, ..., f_{i-1}, e_{i+1}, ..., e_m\}$). It was shown in [DW4] that it is possible to find for every weight function $\eta: E \to \mathbb{R}$ the maximal value of the map $v_\eta: \binom{E}{m} \to \{-\infty\} \cup \mathbb{R}: B \mapsto v(B) + \sum_{e \in B} \eta(e)$ by this algorithm if and only if v satisfies our axiom:

> For all $B, B' \in \binom{E}{m}$ and every $f \in B' \backslash B$ there exists some $e \in B \backslash B'$, now with
>
> $$v(B) + v(B') \leqslant v(\{f\} \cup (B \backslash \{e\})) + v(\{e\} \cup (B' \backslash \{f\})).$$

In this article we study valuated matroids systematically from a more algebraic point of view. In Section 1 we show how a valuation v of a matroid M, defined on E, induces valuations on the *minors* of M and—in case of finite E—also on its *dual.* In Section 2 we study *projective equivalence* of valuations where two valuations v, $v': \binom{E}{m} \to \bar{\Gamma}$ are called projectively equivalent if and only if there exists some $\alpha \in \Gamma$ and some map $\eta: E \to \Gamma$ such that for all $B = \{b_1, ..., b_m\} \in \binom{E}{m}$ we have

$$v'(B) = \alpha \cdot \prod_{i=1}^{m} \eta(b_i) \cdot v(B)$$

and we discuss in particular for a given valuated matroid (E, v) the various combinatorial geometries, defined on E, whose bases are the sets

$$\mathscr{B}^{v'} := \left\{ B \in \binom{E}{m} \middle| v'(B) \geqslant v'(B') \text{ for all } B' \in \binom{E}{m} \right\},$$

where v' varies over all valuations, projectively equivalent to v, the "*residue class geometries of* (M, v)." In Section 3 we will compare valuations of two matroids, defined on the same set E, which differ only by a single base.

Fuzzy rings and matroids with coefficients in a fuzzy ring have been introduced in [D1] as a concept which simultaneously unifies and generalizes ordinary, binary, ternary, regular, oriented, etc., matroids. In Section 4 we establish a one-to-one correspondence between valuated matroids with values in some Γ as above and matroids with coefficients in some appropriately defined fuzzy ring K_Γ—a correspondence which in view of [D1] allows us in particular to define valuated matroids and their duals without any finiteness restrictions.

So far, valuated and oriented matroids appear to have many properties in common. However, a remarkable difference is the fact that every matroid has at least a "trivial" valuation, that is, the valuation v_0 with $v_0(B)=1$ for all bases B. This is nothing but an obvious reformulation of the strong exchange property for bases of matroids. In consequence, every matroid has all the valuations which are projectively equivalent to v_0. Hence the problem is not whether a given matroid has a valuation but whether it is *rigid*, that is, whether all of its valuations are projectively equivalent to v_0. As suggested by the essentially obvious fact that finite fields admit only trivial valuations we prove in the last section that binary matroids as well as finite projective spaces are rigid.

It should be mentioned here that Sections 1, 2, and 3 are rather elementary and can be understood easily by any reader familiar with the fundamental aspects of matroid theory. However, Sections 4 and 5 demand some familiarity with the machinery, developed in [D1, DW1, DW2, and W1, W2], concerning matroids with coefficients and the Tutte group of a matroid.

1. Valuations of Combinatorial Geometries

For a prime number $p \in \mathbb{N}$ define $v_p\colon \mathbb{Q} \to \mathbb{Q}^+ \cup \{0\}$ by

$$v_p(0) := 0, \tag{1.1a}$$

$$v_p\left(\frac{l}{k} \cdot p^n\right) := p^{-n}, \qquad \text{where} \quad n \in \mathbb{Z} \text{ and } l, k \in \mathbb{Z} \setminus p \cdot \mathbb{Z}. \tag{1.1b}$$

v_p is a non-archimedean valuation of $\mathbb{Q}$, i.e., v_p satisfies

$$v_p(q) \geqslant 0 \qquad \text{for all} \quad q \in \mathbb{Q} \text{ and } v_p(q)=0 \text{ if and only if } q=0, \tag{1.2a}$$

$$v_p(q_1 \cdot q_2) = v_p(q_1) \cdot v_p(q_2) \qquad \text{for all} \quad q_1, q_2 \in \mathbb{Q}, \tag{1.2b}$$

$$v_p(q_1 + q_2) \leqslant \max(v_p(q_1), v_p(q_2)) \qquad \text{for all} \quad q_1, q_2 \in \mathbb{Q}. \tag{1.2c}$$

In particular, we have

$$v_p(q) = v_p(-q) \qquad \text{for all} \quad q \in \mathbb{Q}. \tag{1.2d}$$

Assume $m \in \mathbb{N}$. Then for any $2m$ vectors $e_0, ..., e_m, f_2, ..., f_m \in \mathbb{Q}^m$ we have the well known identity of Grassmann,

$$\sum_{i=0}^{m} (-1)^i \cdot \det(e_0, ..., \hat{e}_i, ..., e_m) \cdot \det(e_i, f_2, ..., f_m) = 0.^1 \tag{1.3}$$

(For a proof see [DW3, (6.1).]

[1] As usual, the symbol $(e_0, ..., \hat{e}_i, ..., e_m)$ denotes the sequence $(e_0, ..., e_{i-1}, e_{i+1}, ..., e_m)$ one obtains from $(e_0, ..., e_m)$ by deleting e_i.

Hence (1.2b), (1.2c), and (1.2d) imply that for $e_0, ..., e_m, f_2, ..., f_m$ as above there exists some i with $1 \leqslant i \leqslant m$ and

$$v_p(\det(e_1, ..., e_m)) \cdot v_p(\det(e_0, f_2, ..., f_m)) \\ \leqslant v_p(\det(e_0, ..., \hat{e}_i, ..., e_m)) \cdot v_p(\det(e_i, f_2, ..., f_m)). \tag{1.4}$$

This observation leads us to the following.

DEFINITION 1.1. Assume E is some set, $m \in \mathbb{N}$ and $\Gamma = (\Gamma, \cdot, \leqslant)$ is a linearly ordered abelian group, i.e., $(\Gamma, \leqslant)$ is totally ordered and satisfies the axiom

For $\alpha, \beta, \gamma \in \Gamma$ with $\alpha < \beta$ we have $\alpha \cdot \gamma < \beta \cdot \gamma$.

Put $\bar{\Gamma} := \Gamma \cup \{0\}$, define $\alpha \cdot 0 = 0 \cdot \alpha := 0$ for all $\alpha \in \bar{\Gamma}$ and $0 < \alpha$ for all $\alpha \in \Gamma$.

(i) A map $v: E^m \to \bar{\Gamma}$ defines a *valuated matroid* $M_v = (E, v)$ on E *of rank m with values in Γ*, if the following properties are satisfied:

(V0) There exist $e_1, ..., e_m \in E$ with $v(e_1, ..., e_m) \neq 0$.

(V1) For $e_1, ..., e_m \in E$ and every permutation $\tau \in \Sigma_m$ we have $v(e_1, ..., e_m) = v(e_{\tau(1)}, ..., e_{\tau(m)})$.

Furthermore, in case the cardinality $\# E'$ of $E' := \{e_1, ..., e_m\}$ satisfies $\# E' < m$ we have $v(e_1, ..., e_m) = 0$.

(V2) For $e_0, ..., e_m, f_2, ..., f_m \in E$ there exists some i with $1 \leqslant i \leqslant m$ and $v(e_1, ..., e_m) \cdot v(e_0, f_2, ..., f_m) \leqslant v(e_0, ..., \hat{e}_i, ..., e_m) \cdot v(e_i, f_2, ..., f_m)$.

If (V0), (V1), (V2) are satisfied, v is called a *valuation* of M_v.

(ii) Two valuations $v_1, v_2: E^m \to \bar{\Gamma}$ are said to define the same valuated matroid M, i.e., $M_{v_1} = M_{v_2}$, if there exists some $\alpha \in \Gamma$ with $v_1 \equiv \alpha \cdot v_2$.

In this case v_1 and v_2 are said to be *equivalent*.

(iii) $\{e_1, ..., e_m\}$ is called a *base* of M_v, if $v(e_1, ..., e_m) \neq 0$.

CONVENTION. For $A = \{a_1, ..., a_m\} \in \binom{E}{m} := \{A \subseteq E \mid \# A = m\}$ we write $v(A) := v(a_1, ..., a_m)$. (V1) implies that $v(A)$ is well defined.

Remarks. (i) By (V2) it is clear that the bases of a valuated matroid are also the bases of a combinatorial geometry (or matroid) in the ordinary sense. This (ordinary) combinatorial geometry is called the underlying combinatorial geometry of the valuated matroid, given by v, and is denoted by $\underline{M_v}$.

Vice versa, if M is a combinatorial geometry of rank m, defined on E, then any map v from E^m into some $\bar{\Gamma} = \Gamma \cup \{0\}$ as above which satisfies the conditions (V0), (V1), (V2) is called a valuation of M if for all $e_1, ..., e_m \in E$ one has $v(e_1, ..., e_m) \neq 0$ if and only if $\{e_1, ..., e_m\}$ is a base of M.

(ii) If p is a prime number, v_p is defined as above and $E \subseteq \mathbb{Q}^m$ spans $\mathbb{Q}^m$, then $v := v_p \circ \det: E^m \to \mathbb{Q}^+ \cup \{0\}$ is a valuation of the combinatorial geometry defined on E by linear (in)dependence over $\mathbb{Q}$.

(iii) If Γ' is a subgroup of a linearly ordered abelian group Γ, then any valuated matroid M_v, defined on E, with values in Γ' is also a valuated matroid with values in Γ.

(iv) A valuation $v: E^m \to \bar{\Gamma}$ is called *trivial*, if it is equivalent to a valuation with values in $\{1\} \leqslant \Gamma$.

Every combinatorial geometry M of rank m, defined on E, has a trivial valuation $v: E^m \to \overline{\{1\}} (\subseteq \bar{\Gamma})$,

$$v(e_1, ..., e_m) = \begin{cases} 1 & \text{if } \{e_1, ..., e_m\} \text{ is a base in } M \\ 0 & \text{otherwise.} \end{cases}$$

In this case (V2) states nothing but the well-known strong exchange property for bases of M:

> "For two bases B_1, B_2 of M and $e \in B_1 \backslash B_2$ there exists some $f \in B_2 \backslash B_1$ such that $(B_1 \backslash \{e\}) \cup \{f\}$ and $(B_2 \backslash \{f\}) \cup \{e\}$ are bases of M."

In the sequel we assume that Γ and $\bar{\Gamma} = \Gamma \cup \{0\}$ are as in Definition 1.1.

Let M denote a combinatorial geometry defined on E and of rank $m < \infty$ with $\mathscr{B} = \mathscr{B}_M$ as its set of bases. By Definition 1.1 the valuations of M with values in Γ correspond naturally and in a one-to-one fashion to all maps $v: \mathscr{B} \to \Gamma$ with the following property:

(V) For $B_1, B_2 \in \mathscr{B}$ and $e \in B_1 \backslash B_2$ there exists some $f \in B_2 \backslash B_1$ with $B_1' := (B_1 \backslash \{e\}) \cup \{f\} \in \mathscr{B}$, $B_2' := (B_2 \backslash \{f\}) \cup \{e\} \in \mathscr{B}$ and $v(B_1) \cdot v(B_2) \leqslant v(B_1') \cdot v(B_2')$.

Therefore a map $v: \mathscr{B} \to \Gamma$ satisfying (V) will also be called a valuation of M.

We will now show how $v: \mathscr{B} \to \Gamma$ induces valuations on the minors of M. Let ρ_M denote the rank function of M.

PROPOSITION 1.2. *Assume $F \subseteq E$, put $E' := E \backslash F$ and let k denote the rank of the matroid $M' := M \backslash F = M | E'$. Choose $I := \{f_{k+1}, ..., f_m\} \subseteq F$*

such that $\rho_M(E' \cup I) = m$. If $\mathscr{B}'$ denotes the set of bases of M', then $v_I: \mathscr{B}' \to \Gamma$ defined by $v_I(B) := v(B \cup I)$ is a valuation of M'.

If $w: \mathscr{B} \to \Gamma$ is equivalent to v, then v_I and w_I are also equivalent.

Finally, if $I' := \{f'_{k+1}, ..., f'_m\} \subseteq F$ also satisfies $\rho_M(E' \cup I') = m$, then v_I and $v_{I'}$ are equivalent, too.

Proof. The first assertion follows at once from the fact that for $B \subseteq E'$ we have $B \in \mathscr{B}'$ if and only if $B \cup I \in \mathscr{B}$, while the second one is trivial.

To show the last assertion we may assume $k = m - 1$ and therefore, say, $I = \{e\}$ and $I' = \{f\}$ with $e \neq f$.

For $B_1, B_2 \in \mathscr{B}'$ we have to prove

$$v(B_1 \cup \{e\}) \cdot v(B_1 \cup \{f\})^{-1} = v(B_2 \cup \{e\}) \cdot v(B_2 \cup \{f\})^{-1}.$$

By symmetry we are done once it is shown

$$v(B_1 \cup \{e\}) \cdot v(B_2 \cup \{f\}) \leqslant v(B_1 \cup \{f\}) \cdot v(B_2 \cup \{e\}).$$

But this follows from (V), because for $b \in B_2 \cup \{f\}$ with $B_1 \cup \{b\} \in \mathscr{B}$ one has $b = f$. ∎

PROPOSITION 1.3. *Assume $F \subseteq E$, put $E' := E \backslash F$ and let k denote the rank of the matroid $M' := M/F$. Choose $I := \{f_{k+1}, ..., f_m\} \subseteq F$ such that $\rho_M(I) = m - k = \rho_M(F)$. If $\mathscr{B}'$ denotes the set of bases of M', then $v_I: \mathscr{B}' \to \Gamma$ defined by $v_I(B) := v(B \cup I)$ is a valuation of M'.*

If $w: \mathscr{B} \to \Gamma$ is equivalent to v, then v_I and w_I are also equivalent.

Finally, if $I' := \{f'_{k+1}, ..., f'_m\} \subseteq F$ satisfies $\rho_M(I') = m - k$, too, then v_I and $v_{I'}$ are equivalent as well.

Proof. Again the first assertion follows from the fact that for $B \subseteq E'$ we have $B \in \mathscr{B}'$ if and only if $B \cup I \in \mathscr{B}$, while the second one is trivial.

The last assertion follows once it is shown that for $B_1, B_2 \in \mathscr{B}'$ with $\#(B_1 \backslash B_2) = \#(B_2 \backslash B_1) = 1$, say $B_1 \backslash B_2 = \{e\}$, $B_2 \backslash B_1 = \{f\}$, we have

$$v(B_1 \cup I) \cdot v(B_2 \cup I') \leqslant v(B_2 \cup I) \cdot v(B_1 \cup I').$$

But this follows from (V), because for $b \in (B_2 \cup I') \backslash (B_1 \cup I)$ with $((B_1 \cup I) \backslash \{e\}) \cup \{b\} \in \mathscr{B}$ we must have $b = f$. ∎

At the end of this section we show

PROPOSITION 1.4. *If $\# E = m + n$ is finite and $\mathscr{B}^* := \{E \backslash B \mid B \in \mathscr{B}\}$ denotes the set of bases of the dual matroid M^* of M, then $v^*: \mathscr{B}^* \to \Gamma$ defined by $v^*(E \backslash B) := v(B)$ is a valuation of M^*.*

If $w: \mathscr{B} \to \Gamma$ is equivalent to v, then v^* and w^* are also equivalent.

Proof. Assume $B_1, B_2 \in \mathscr{B}$ and put $B_1^* := E \backslash B_1$, $B_2^* := E \backslash B_2$. For $e \in B_1^* \backslash B_2^* = B_2 \backslash B_1$ there exists $f \in B_1 \backslash B_2 = B_2^* \backslash B_1^*$ with $B_2' := (B_2 \backslash \{e\}) \cup \{f\} \in \mathscr{B}$, $B_1' := (B_1 \backslash \{f\}) \cup \{e\} \in \mathscr{B}$ and $v(B_1) \cdot v(B_2) \leqslant v(B_1') \cdot v(B_2')$ by (V). This means $E \backslash B_1' = (B_1^* \backslash \{e\}) \cup \{f\} \in \mathscr{B}^*$, $E \backslash B_2' = (B_2^* \backslash \{f\}) \cup \{e\} \in \mathscr{B}^*$ and

$$v^*(E \backslash B_1) \cdot v^*(E \backslash B_2) \leqslant v^*(E \backslash B_1') \cdot v^*(E \backslash B_2').$$

The last assertion is trivial. ∎

2. Similarity of Valuations

In the sequel we assume that M is a combinatorial geometry defined on E and of rank $m < \infty$ with $\mathscr{B} = \mathscr{B}_M$ as its set of bases and that $\Gamma = (\Gamma, \cdot, \leqslant)$ is a linearly ordered abelian group.

The following trivial lemma shows how we can construct many valuations of M in an obvious way, starting from a given one.

Lemma and Definition 2.1. *If $v: E^m \to \bar{\Gamma}$ is a valuation of M, if $\alpha \in \Gamma$ and $\eta: E \to \Gamma$ is a map, then $w := v[\alpha, \eta]: E^m \to \bar{\Gamma}$ defined by*

$$w(e_1, ..., e_m) := \alpha \cdot \prod_{i=1}^{m} \eta(e_i) \cdot v(e_1, ..., e_m) \tag{2.1}$$

is also a valuation of M. Hence, two valuations v, w of M will be called projectively equivalent or similar, if $w = v[\alpha, \eta]$ for some $\alpha \in \Gamma$ and some map $\eta: E \to \Gamma$.

Remarks. (i) Projective equivalence is clearly an equivalence relation: $w = v[\alpha, \eta]$ and $u = w[\beta, \eta']$ imply $v = w[\alpha^{-1}, \eta^{-1}]$ and $u = v[\alpha \cdot \beta, \eta \cdot \eta']$ where $\eta^{-1}: E \to \Gamma$ is, of course, given by $\eta^{-1}(e) := \eta(e)^{-1}$ for $e \in E$.

(ii) Equivalent valuations are also projectively equivalent, while the converse is not true.

Definition 2.2. (i) A valuation v is said to be *essentially trivial* if v is similar to one and thus to all trivial valuations.

(ii) M is said to be *rigid*, if every valuation of M with values in any linearly ordered abelian group Γ is essentially trivial.

The following trivial, but very useful result follows immediately from these definitions.

Lemma 2.3. *A valuation $v: E^m \to \bar{\Gamma}$ of M is essentially trivial if and only* *if there exists some $\alpha \in \Gamma$ and some map $\eta: E \to \Gamma$ such that for all $e_1, \ldots, e_m \in E$ we have*

$$v(e_1, \ldots, e_m) = \begin{cases} \alpha \cdot \prod_{i=1}^m \eta(e_i) & \text{if } \{e_1, \ldots, e_m\} \in \mathscr{B}_M \\ 0 & \textit{otherwise.} \end{cases}$$

This lemma in turn yields at once the following criterion for a valuation not to be essentially trivial.

PROPOSITION 2.4. *Assume $v: E^m \to \bar{\Gamma}$ is a valuation of M such that there exist $k \in \mathbb{N}$ and bases $B_1, \ldots, B_k, B'_1, \ldots, B'_k \in \mathscr{B}_M$ with*

$$\#\{i \mid e \in B_i\} = \#\{i \mid e \in B'_i\} \qquad \textit{for all} \quad e \in E$$

and

$$\prod_{i=1}^{k} v(B_i) \neq \prod_{i=1}^{k} v(B'_i).$$

Then v is not essentially trivial.

EXAMPLE 2.5. Using Proposition 2.4 we show that the uniform matroid $U_{2,4}$ of rank 2 with 4 elements is not rigid.

Put $a := \binom{2}{1}$, $b := \binom{2}{3}$, $c := \binom{1}{1}$, $d := \binom{3}{1}$, $E := \{a, b, c, d\} \subseteq \mathbb{Q}^2$ and define $v: E^2 \to \mathbb{Q}^+ \cup \{0\}$ by

$$v(e_1, e_2) := v_2(\det(e_1, e_2)).$$

We know (cf. Remark (ii)) following Definition 1.1) that v is a valuation of the matroid $M = U_{2,4}$ on E defined by linear (in)dependence.

Now put $k := 2$, $B_1 := \{a, b\}$, $B_2 := \{c, d\}$, $B'_1 := \{a, c\}$, $B'_2 := \{b, d\}$. Then we have $E = B_1 \cup B_2 = B'_1 \cup B'_2$, but

$$v(a, b) \cdot v(c, d) = \tfrac{1}{4} \cdot \tfrac{1}{2} \neq 1 = v(a, c) \cdot v(b, d);$$

thus v is not essentially trivial.

Remark 2.6. In view of [W2, Theorem 6.5] it is not accidental that $U_{2,4}$ is non-rigid. Indeed, it follows from this result that all matroids would be rigid if $U_{2,4}$ were rigid.

In Section 5 we see that all binary matroids as well as all finite projective spaces of dimension at least two are rigid.

In Section 1 we saw how a valuation of M induces valuations on its minors and in case of finite E also a valuation on its dual M^*.

We show now that two similar valuations v, w defined on M induce similar valuations on its minors and for finite E also on M^*.

DEFINITION 2.7. If E is finite and $\eta: E \to \Gamma$ is a map, then the *weight* $P(\eta)$ of η is defined by

$$P(\eta) := \prod_{e \in E} \eta(e). \tag{2.2}$$

PROPOSITION 2.8. *Assume $v, w: E^m \to \bar{\Gamma}$ are similar, say $w = v[\alpha, \eta]$.*

(i) *If E', M', and I are as in Proposition 1.2 or as in Proposition 1.3, then v_I and w_I are also similar. More precisely, we have*

$$w_I = v_I\left[\alpha \cdot \prod_{f \in I} \eta(f), \eta|_{E'}\right].$$

(ii) *If $\# E = m + n < \infty$, then $v^*, w^*: E^n \to \bar{\Gamma}$ are also similar. More precisely, we have*

$$w^* = v^*[\alpha \cdot P(\eta), \eta^{-1}].$$

Proof. (i) is obvious.

(ii) If $B_0 = \{e_1, ..., e_n\}$ is a base of M^* and $\{f_1, ..., f_m\} = E \backslash B_0$, then we have

$$\begin{aligned} w^*(e_1, ..., e_n) &= w(f_1, ..., f_m) \\ &= \alpha \cdot \prod_{i=1}^{m} \eta(f_i) \cdot v(f_1, ..., f_m) \\ &= \alpha \cdot P(\eta) \cdot \prod_{i=1}^{n} \eta(e_i)^{-1} \cdot v^*(e_1, ..., e_n). \quad \blacksquare \end{aligned}$$

We will now show how a valuation of a combinatorial geometry M possibly—and in any case for finite E—induces in a canonical way a new combinatorial geometry defined on E which is a weak image of M.

PROPOSITION 2.9. *Assume that $v: E^m \to \bar{\Gamma}$ is a valuation of M and put*

$$\mathscr{B}^v := \{\{e_1, ..., e_m\} \subseteq E \mid v(e_1, ..., e_m) \geqslant v(f_1, ..., f_m) \text{ for all } f_1, ..., f_m \in E\}.$$

(i) *If $\mathscr{B}^v \neq \varnothing$, in particular, if E is finite, then $\mathscr{B}^v$ is the set of bases of some combinatorial geometry M^v defined on E, called the v-image of M.*

(ii) *The identity map defined on E is a weak homomorphism from M to M^v, if M^v is defined, i.e., if $\mathscr{B}^v \neq \varnothing$.*

(iii) *Assume M^v is defined. Then $e \in E$ is a loop in M^v if and only if for every $B \in \mathscr{B}$ with $e \in B$ there exists some $B' \in \mathscr{B}$ with $e \notin B'$ and $v(B) < v(B')$. Similarly, for $e, f \in E$ with $e \neq f$ the set $\{e, f\}$ is a circuit in M^v*

if and only if e, f are no loops in M^v *and for every* $B \in \mathscr{B}$ *with* $\{e, f\} \subseteq B$ *there exists some* $B' \in \mathscr{B}$ *with* $\{e, f\} \nsubseteq B'$ *and* $v(B) < v(B')$.

(iv) *We have* $\mathscr{B} = \mathscr{B}^v$ *and* $M = M^v$ *if and only if the valuation v is trivial.*

Proof. (i) follows directly from (V2), because $B_1, B_2 \in \mathscr{B}^v$, $B'_1, B'_2 \in \mathscr{B}$, and $v(B_1) \cdot v(B_2) \leqslant v(B'_1) \cdot v(B'_2)$ imply $B'_1, B'_2 \in \mathscr{B}^v$.

(ii) follows from the fact that by definition the identity map is a weak homomorphism between two matroids M and M' defined on E of finite rank m with $\mathscr{B}$ and $\mathscr{B}'$ as their sets of bases, respectively, if and only if $\mathscr{B}' \subseteq \mathscr{B}$.

(iii) and (iv) are also obvious. ▌

Remarks. (i) If $v, w\colon E^m \to \bar{\Gamma}$ are equivalent valuations, then we have $\mathscr{B}^v = \mathscr{B}^w$.

(ii) If v and w are similar, then, in general, $\mathscr{B}^v \neq \mathscr{B}^w$. In case $\# E = \infty$ there (may even) exist similar valuations v, w with $\mathscr{B}^v = \varnothing$, $\mathscr{B}^w \neq \varnothing$: Assume p is a prime number, $E = \mathbb{Q}^m$, $v = v_p \circ \det$ and $w = v[1, \eta]$ with $\eta\colon \mathbb{Q}^m \to \mathbb{Q}^+$ given by

$$\eta\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} := \begin{cases} 1 & \text{if } \; a_i = 0 \text{ for all } i \text{ with } 1 \leqslant i \leqslant m \\ (\max\limits_{1 \leqslant i \leqslant m} v_p(a_i))^{-1} & \text{else.} \end{cases}$$

Clearly, we have $\mathscr{B}^v = \varnothing$.

On the other hand, we have

$$w\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\right) = 1$$

and

$$w(e_1, \ldots, e_m) = w(p^{\alpha_1} \cdot e_1, \ldots, p^{\alpha_m} \cdot e_m) \leqslant 1$$

for all $e_1, \ldots, e_m \in E$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}$. This implies

$$\mathscr{B}^w = \{ \{b_1, \ldots, b_m\} \in \tbinom{E}{m} \mid w(b_1, \ldots, b_m) = 1 \} \neq \varnothing.$$

We will now study the various $\mathscr{B}^v$ for a whole class of projectively equivalent valuations $v\colon E^m \to \bar{\Gamma}$ and in particular those $\mathscr{B}^v$ which are maximal.

Clearly, a valuation v of the matroid M is essentially trivial if and only if $\mathscr{B}^w = \mathscr{B}_M$ for some valuation w similar to v.

At first we show the following general result about matroids:

LEMMA 2.10. *Assume that* $B_0 = \{e_1, ..., e_m\}$ *and* $B_1 = \{x_1, ..., x_n, e_{n+1}, ..., e_m\}$ *are bases of the matroid M such that* $B_0 \cap B_1 = \{e_{n+1}, ..., e_m\}$. *Then there exists some permutation* $\tau \in \Sigma_n$ *with* $(B_0 \backslash \{e_i\}) \cup \{x_{\tau(i)}\} \in \mathscr{B}_M$ *for all i with* $1 \leqslant i \leqslant n$.

Proof. We proceed by induction on $n = \#(B_1 \backslash B_0)$.

The cases $n = 0$ and $n = 1$ are trivial.

Now assume $2 \leqslant n \leqslant m$. Then by the strong exchange property for bases of matroids there exists j with $1 \leqslant j \leqslant n$ such that $B_0' := (B_0 \backslash \{e_j\}) \cup \{x_n\} \in \mathscr{B}_M$ and $B_1' := (B_1 \backslash \{x_n\}) \cup \{e_j\} \in \mathscr{B}_M$. Our induction hypothesis applied to B_0, B_1' implies that there exists a bijection τ': $\{1, ..., n\} \backslash \{j\} \to \{1, ..., n-1\}$ with $(B_0 \backslash \{e_i\}) \cup \{x_{\tau'(i)}\} \in \mathscr{B}_M$ for all i with $i \neq j$ and $1 \leqslant i \leqslant n$. Thus $\tau \in \Sigma_n$, defined by $\tau(i) := \tau'(i)$ for $i \neq j$ and $\tau(j) := n$, satisfies what we want. ∎

Next we prove

LEMMA 2.11. *Assume* $v: E^m \to \bar{\Gamma}$ *is a valuation of M and* $B_0 = \{e_1, ..., e_m\} \in \mathscr{B}_M$. *Then we have* $B_0 \in \mathscr{B}^v$ *if and only if*

$$v(e_1, ..., \hat{e}_i, ..., e_m, x) \leqslant v(e_1, ..., e_m) \text{ holds for all } x \in E \text{ and } 1 \leqslant i \leqslant m.$$

Proof. The necessity of our assumption is obvious. Assume that, vice versa, this condition holds. We prove by induction on $n := \#(\{x_1, ..., x_m\} \backslash \{e_1, ..., e_m\})$ that for all $x_1, ..., x_m \in E$ we have $v(x_1, ..., x_m) \leqslant v(e_1, ..., e_m)$. The cases $n = 0$ and $v(x_1, ..., x_m) = 0$ are trivial, while for $n = 1$ we are done by the assumption of our lemma.

Now assume $2 \leqslant n \leqslant m$, say $\#\{x_1, ..., x_n, e_1, ..., e_m\} = n + m$ and $x_k = e_k$ for $n + 1 \leqslant k \leqslant m$. Then by (V2) and our induction hypothesis there exists i with $1 \leqslant i \leqslant n$ and

$$\begin{aligned} & v(x_1, ..., x_n, e_{n+1}, ..., e_m) \cdot v(e_1, ..., e_m) \\ & \leqslant v(e_i, x_2, ..., x_n, e_{n+1}, ..., e_m) \cdot v(x_1, e_1, ..., \hat{e}_i, ..., e_m) \\ & \leqslant v(e_1, ..., e_m)^2. \end{aligned}$$

This means $v(x_1, ..., x_n, e_{n+1}, ..., e_m) \leqslant v(e_1, ..., e_m)$, because $v(e_1, ..., e_m) \neq 0$. ∎

Using the last result we will now show

PROPOSITION 2.12. *Assume* $v: E^m \to \bar{\Gamma}$ *is a valuation of* M *and* $B = \{e_1, ..., e_m\} \in \mathscr{B}_M$. *Define* $\eta_B: E \to \Gamma$ *by*

$$\eta_B(x) := \begin{cases} 1 & \text{if } x \text{ is a loop in } M \\ (\max\limits_{1 \leqslant j \leqslant m} v(e_1, ..., \hat{e}_j, ..., e_m, x))^{-1} \cdot v(e_1, ..., e_m) & \text{else} \end{cases}$$

and put $v_B := v[1, \eta_B]$. *Then the following holds*:

(i) *We have* $B \in \mathscr{B}^{v_B}$, *and for every* $x \in E \setminus B$ *which is not a loop in* M *there exists some* j *with* $1 \leqslant j \leqslant m$ *and* $(B \setminus \{e_j\}) \cup \{x\} \in \mathscr{B}^{v_B}$. *In particular, for any* $B \in \mathscr{B}_M$ *and any valuation* $v: E^m \to \bar{\Gamma}$ *of* M *there exists some valuation* w *similar to* v *with* $B \in \mathscr{B}^w$.

(ii) *If* $B \in \mathscr{B}^v$, *then we have* $\mathscr{B}^v \subseteq \mathscr{B}^{v_B}$, *and we have* $\mathscr{B}^v = \mathscr{B}^{v_B}$ *if and only if* $v = v_B$ *if and only if* $\mathscr{B}^v \cap \{\{e_1, ..., \hat{e}_j, ..., e_m, x\} \mid 1 \leqslant j \leqslant m, x \in E\} = \mathscr{B}^{v_B} \cap \{\{e_1, ..., \hat{e}_j, ..., e_m, x\} \mid 1 \leqslant j \leqslant m, x \in E\}$ *if and only if for every* $x \in E$ *which is not a loop in* M *there exists some* j *with* $1 \leqslant j \leqslant m$ *and* $(B \setminus \{e_j\}) \cup \{x\} \in \mathscr{B}^v$.

Proof. (i) For $1 \leqslant i \leqslant m$ we have $\eta_B(e_i) = 1$. Thus we obtain for all $x \in E$ and $1 \leqslant i \leqslant m$,

$$\begin{aligned} v_B(e_1, ..., \hat{e}_i, ..., e_m, x) &= \eta_B(x) \cdot v(e_1, ..., \hat{e}_i, ..., e_m, x) \\ &\leqslant v(e_1, ..., e_m) \\ &= v_B(e_1, ..., e_m), \end{aligned}$$

and if x is not a loop, we have $v_B(e_1, ..., \hat{e}_j, ..., e_m, x) = v_B(e_1, ..., e_m)$ for a suitable j with $1 \leqslant j \leqslant m$. Thus we are done, since by Lemma 2.11 we have $B \in \mathscr{B}^{v_B}$.

(ii) follows because of $B \in \mathscr{B}^{v_B}$ immediately from the facts that by assumption $\eta_B(x) \geqslant 1$ for all $x \in E$ and $\eta_B(x) = 1$ if and only if x is a loop or there exists j with $(B \setminus \{e_j\}) \cup \{x\} \in \mathscr{B}^v$. ∎

If v' is similar to v and if we can choose $\alpha' \in \Gamma$ and $\eta': E \to \Gamma$ with $v' = v[\alpha', \eta']$ such that η' is constant on B, then v'_B and v_B are obviously equivalent, so, replacing v by v', we see that $B \in \mathscr{B}^{v'}$ implies $\mathscr{B}^{v'} \subseteq \mathscr{B}^{v_B}$ and $\mathscr{B}^{v'} = \mathscr{B}^{v_B}$ if and only if v' and v_B are equivalent. Still, once we consider valuations $w = v[\alpha, \eta]$ of M, for which η is not constant on B, we may find some such valuations w for which $\mathscr{B}^w$ properly contains $\mathscr{B}^{v_B}$. To discuss this possibility more thoroughly let us assume that v is a valuation of M with $\mathscr{B}^v \neq \varnothing$ and let us choose some fixed $B_0 = \{e_1, ..., e_m\} \in \mathscr{B}^v$. We define two bipartite graphs $G_1=G_1(B_0)$ and $G_2=G_2(B_0, v)$ with $E=B_0 \cup (E\backslash B_0)$ as their set of vertices and

$$K_1=K_1(B_0):=\{\{e_i, x\} \mid (B_0\backslash\{e_i\}) \cup \{x\} \in \mathscr{B}\},$$
$$K_2=K_2(B_0, v):=\{\{e_i, x\} \mid v(e_1, ..., \hat{e}_i, ..., e_m, x)=v(B_0)\}$$

as their sets of edges, respectively. Clearly, we have $K_2 \subseteq K_1$.

If S denotes the set of loops of M, then the bases of M are exactly the bases of the restriction $M':=M\backslash S$, and therefore the valuations of M correspond naturally and in a one-to-one fashion to those of M'. Thus from now on we may assume that M does not contain any loops, which implies that the number of connected components of $G_1(B_0)$ is finite and that in view of Proposition 2.12 the same holds for $G_2(B_0, v)$ in case $v=v_{B_0}$.

Let $z_1=z_1(B_0)$ and $z_2=z_2(B_0, v)$ denote the number of connected components of $G_1(B_0)$ and $G_2(B_0, v)$, respectively; so we have $z_1 \leqslant z_2 \leqslant \infty$, but $z_1<\infty$.

We will show in the sequel that $\mathscr{B}^v$ is maximal among all $\mathscr{B}^w$ with w similar to v if $z_1(B_0)=z_2(B_0, v)$. Moreover, the converse will also be shown to be true, if the following finiteness condition (F) is satisfied:

(F) For all $B \in \mathscr{B}$ and all w similar to v the set $\{w(B) \cdot w(B')^{-1} \mid B' \in \mathscr{B}$ and $w(B') \leqslant w(B)\}$ is well ordered.

For example, this condition (F) is satisfied for all finite sets E or in case $\Gamma \cong \mathbb{Z}$, say $\Gamma=\{p^n \mid n \in \mathbb{Z}\}$ for some prime number p.

Let $E_1, ..., E_{z_1}$ denote the sets of vertices of the z_1 distinct connected components of $G_1(B_0)$. We will now relate these connected components to the connected components of the matroid M as defined in [We, Chap. 5]: if $e, f \in E$ are distinct, then e and f belong to the same connected component of M if and only if there exists some circuit C in M with $\{e, f\} \subseteq C$.

Proposition 2.13. *The matroids $M \mid E_1, ..., M \mid E_{z_1}$ are exactly the connected components of the matroid M. In particular, z_1 and $E_1, ..., E_{z_1}$ do not depend on the fixed base $B_0 \in \mathscr{B}^v$.*

Proof. If $(B_0\backslash\{e_j\}) \cup \{x\} \in \mathscr{B}$ for some j with $1 \leqslant j \leqslant m$ and $x \in E\backslash B_0$, then e_j and x lie in the same connected component of the matroid M. Thus the proposition is proved once it is shown that $\#(B_0 \cap E_i)=\rho_M(E_i)$ for all i with $1 \leqslant i \leqslant z_1$, where ρ_M denotes the rank function of M. Otherwise there exists some base $B_1 \in \mathscr{B}$ and some i with $1 \leqslant i \leqslant z_1$ and $\#(B_0 \cap E_i)<\#(B_1 \cap E_i)=\rho_M(E_i)$. By a repeated application of the base exchange axiom we see that there exists some base $B_2 \in \mathscr{B}$ with $B_1 \cap E_i = B_2 \cap E_i$ and $B_2 \backslash E_i \subseteq B_0$. Then we have

$$\begin{aligned}\#(B_0 \cap (E_i \cup B_2)) &= \#(B_0 \cap E_i) + \#(B_2 \backslash E_i) \\ &< \#(B_2 \cap E_i) + \#(B_2 \backslash E_i) = m;\end{aligned}$$

this means $B_0 \backslash (E_i \cup B_2) \neq \varnothing$. Choose some $e \in B_0 \backslash (E_i \cup B_2)$. Then there exists some $f \in B_2 \backslash B_0$ with $(B_0 \backslash \{e\}) \cup \{f\} \in \mathscr{B}$; that is $\{e, f\} \in K_1(B_0)$. Furthermore, $B_2 \backslash E_i \subseteq B_0$ implies $f \in E_i$. But this contradicts $\{e, f\} \in K_1(B_0)$ and $e \notin E_i$. ▌

Now we can show

PROPOSITION 2.14. *Assume $z_1 = z_2(B_0, v)$. Then for all w similar to v with $\mathscr{B}^v \subseteq \mathscr{B}^w$ the valuations v and w are equivalent. In particular, $\mathscr{B}^v$ is maximal among all $\mathscr{B}^w$ with w similar to v.*

Proof. Assume $w = v[\alpha, \eta]$ for some $\alpha \in \Gamma$ and some map $\eta: E \to \Gamma$ such that $\mathscr{B}^v \subseteq \mathscr{B}^w$. We may assume $w(B_0) = v(B_0)$; this means

$$\alpha = \prod_{i=1}^{m} \eta^{-1}(e_i).$$

For every $\{e_i, x\} \in K_2(B_0, v)$ we have $(B_0 \backslash \{e_i\}) \cup \{x\} \in \mathscr{B}^v \subseteq \mathscr{B}^w$ and therefore

$$\begin{aligned}v(e_1, ..., \hat{e}_i, ..., e_m, x) = v(B_0) &= w(B_0) \\ &= w(e_1, ..., \hat{e}_i, ..., e_m, x) \\ &= \alpha \cdot \prod_{j \neq i} \eta(e_j) \cdot \eta(x) \cdot v(e_1, ..., \hat{e}_i, ..., e_m, x) \\ &= \eta(e_i)^{-1} \cdot \eta(x) \cdot v(e_1, ..., \hat{e}_i, ..., e_m, x);\end{aligned}$$

that is $\eta(x) = \eta(e_i)$. Thus by the assumption of our proposition and by Proposition 2.13 we have $\eta(x) = \eta(y)$ for all $x, y \in E$ which lie in one and the same connected component of the matroid M. Since $\#(B_0 \cap E_i) = \#(B \cap E_i)$ for every $B \in \mathscr{B}$ and every connected component $M | E_i$ of M, we obtain

$$w(B) = \prod_{i=1}^{m} \eta^{-1}(e_i) \cdot \prod_{b \in B} \eta(b) \cdot v(B) = v(B)$$

for all $B \in \mathscr{B}$. ▌

If condition (F) is satisfied, the next result yields in case $z_1 < z_2(B_0, v) < \infty$ a method to construct some valuation $w: E^m \to \bar{\Gamma}$ similar to v with $\mathscr{B}^v \subsetneq \mathscr{B}^w$ and $z_2(B_0, w) < z_2(B_0, v)$. Thus a repeated application of this process starting with v_{B_0} instead of v in case $z_2(B_0, v) = \infty$ finally yields a valuation v_0 similar to v such that $z_1 = z_2(B_0, v_0)$ and $\mathscr{B}^{v_0}$ is maximal among all $\mathscr{B}^w$ with w similar to v.

PROPOSITION 2.15. *Assume that condition* (F) *is satisfied and* $z_1 < z_2(B_0, v)$. *Choose some* $\{e_j, x_0\} \in K_1(B_0) \backslash K_2(B_0, v)$ *such that* e_j *and* x_0 *lie in two distinct connected components of* $G_2(B_0, v)$ *and* $v(e_1, ..., \hat{e}_j, ..., e_m, x_0)$ *is maximal with respect to this property which is possible in view of* (F). *Define* $\eta: E \to \Gamma$ *by*

$$\eta(e) := \begin{cases} v(B_0) \cdot v(e_1, ..., \hat{e}_j, ..., e_m, x_0)^{-1} \\ \qquad \textit{if } e \textit{ and } x_0 \textit{ lie in the same connected component of } G_2(B_0, v) \\ 1 \qquad \textit{else} \end{cases}$$

and put $w := v[\prod_{i=1}^{m} \eta^{-1}(e_i), \eta]$.

Then we have $\mathscr{B}^v \subseteq \mathscr{B}^w$ *and* $(B_0 \backslash \{e_j\}) \cup \{x_0\} \in \mathscr{B}^w \backslash \mathscr{B}^v$. *In particular,* $\mathscr{B}^v$ *is not maximal among all* $\mathscr{B}^w$ *with* w *similar to* v.

Moreover, we have $K_2(B_0, v) \subsetneq K_2(B_0, w)$ *and* e_j *and* x_0 *are connected in* $G_2(B_0, w)$.

Proof. By the definition of w we have $w(B_0) = v(B_0)$. Now assume $1 \leqslant i \leqslant m$ and $x \in E \backslash B_0$. We want to show that $w(e_1, ..., \hat{e}_i, ..., e_m, x) \leqslant v(B_0)$.

If $\{e_i, x\} \in K_2(B_0, v)$, then we have $\eta(x) = \eta(e_i)$ and therefore

$$w(e_1, ..., \hat{e}_i, ..., e_m, x) = \eta(e_i)^{-1} \cdot \eta(x) \cdot v(e_1, ..., \hat{e}_i, ..., e_m, x) = v(B_0).$$

If $\{e_i, x\} \in K_1(B_0) \backslash K_2(B_0, v)$, then we obtain

$$w(e_1, ..., \hat{e}_i, ..., e_m, x) = \eta(e_i)^{-1} \cdot \eta(x) \cdot v(e_1, ..., \hat{e}_i, ..., e_m, x) \leqslant v(B_0)$$

which is clear if e_i and x belong to the same connected component of $G_2(B_0, v)$ but holds also otherwise, because then $\eta(e_i)^{-1} \leqslant 1$, $\eta(x) \leqslant v(B_0) \cdot v(e_1, ..., \hat{e}_j, ..., e_m, x_0)^{-1}$ and our choice of e_j and x_0 together imply

$$\begin{aligned} &\eta(e_i)^{-1} \cdot \eta(x) \cdot v(e_1, ..., \hat{e}_i, ..., e_m, x) \\ &\quad \leqslant v(B_0) \cdot v(e_1, ..., \hat{e}_j, ..., e_m, x_0)^{-1} \cdot v(e_1, ..., \hat{e}_i, ..., e_m, x) \\ &\quad \leqslant v(B_0). \end{aligned}$$

For $\{e_i, x\} \notin K_1(B_0)$ we have

$$w(e_1, ..., \hat{e}_i, ..., e_m, x) = v(e_1, ..., \hat{e}_i, ..., e_m, x) = 0.$$

Thus by Lemma 2.11 we have $B_0 \in \mathscr{B}^w$ and

$$\mathscr{B}^w = \{B \in \mathscr{B} \mid w(B) = v(B_0)\}.$$

Now assume $B \in \mathscr{B}^v \setminus \{B_0\}$, say $B = \{x_1, ..., x_n, e_{n+1}, ..., e_m\}$ and $B_0 \cap B = \{e_{n+1}, ..., e_m\}$. Then by Lemma 2.10 applied to M^v there exists $\tau \in \Sigma_n$ with $(B_0 \setminus \{e_i\}) \cup \{x_{\tau(i)}\} \in \mathscr{B}^v$ for all i with $1 \leqslant i \leqslant n$. By the definition of η this means $\eta(e_i) = \eta(x_{\tau(i)})$ for $1 \leqslant i \leqslant n$ and therefore

$$\begin{aligned} &w(x_1, ..., x_n, e_{n+1}, ..., e_m) \\ &= \prod_{i=1}^{n} (\eta(e_i)^{-1} \cdot \eta(x_{\tau(i)})) \cdot v(x_1, ..., x_n, e_{n+1}, ..., e_m) \\ &= v(B_0). \end{aligned}$$

Thus we have $\mathscr{B}^v \subseteq \mathscr{B}^w$.

Moreover, we have $\eta(e_j) = 1$, $\eta(x_0) = v(B_0) \cdot v(e_1, ..., \hat{e}_j, ..., e_m, x_0)^{-1}$ and thus

$$w(e_1, ..., \hat{e}_j, ..., e_m, x_0) = \eta(e_j)^{-1} \cdot \eta(x_0) \cdot v(e_1, ..., \hat{e}_j, ..., e_m, x_0) = v(B_0),$$

that is $(B_0 \setminus \{e_j\}) \cup \{x_0\} \in \mathscr{B}^w \setminus \mathscr{B}^v$.

The last assertion follows from $\mathscr{B}^v \subseteq \mathscr{B}^w$ and $\{e_j, x_0\} \in K_2(B_0, w)$. ∎

Together Propositions 2.12–2.15 yield

THEOREM 2.16. *Assume the matroid M has no loops, $v: E^m \to \bar{\Gamma}$ is a valuation of M and M and v satisfy condition* (F). *Then the following three statements are equivalent:*

(i) *$\mathscr{B}^v$ is maximal among all $\mathscr{B}^w$ with w similar to v.*

(ii) *$\mathscr{B}^v \neq \varnothing$ and $z_2(B_0, v)$ equals the number of connected components of M for all $B_0 \in \mathscr{B}^v$.*

(iii) *There exists some $B_0 \in \mathscr{B}^v$ such that $z_2(B_0, v)$ equals the number of connected components of M.*

Next we show

PROPOSITION 2.17. *Assume $v: E^m \to \bar{\Gamma}$ is a valuation of M and $B_0 \in \mathscr{B}$, $e_0 \in E \setminus B_0$. Define $\eta: E \to \Gamma$ by*

$$\eta(e) := 1 \qquad \textit{if} \quad e = e_0 \textit{ or } e \textit{ is a loop in } M,$$

$$\eta(b) := \begin{cases} 1 & \textit{for} \quad b \in B_0, (B_0 \backslash \{b\}) \cup \{e_0\} \notin \mathscr{B} \\ v(B_0)^{-1} \cdot v((B_0 \backslash \{b\}) \cup \{e_0\}) & \\ & \textit{for} \quad b \in B_0, (B_0 \backslash \{b\}) \cup \{e_0\} \in \mathscr{B}, \end{cases}$$

$$\eta(e) := v(B_0) \cdot (\max_{b \in B_0} \{\eta(b)^{-1} \cdot v((B_0 \backslash \{b\}) \cup \{e\})\})^{-1}$$

$$\textit{if} \quad e \in E \backslash (B_0 \cup \{e_0\}) \textit{ is not a loop.}$$

Furthermore, put

$$v_{B_0, e_0} := v\left[\prod_{b \in B_0} \eta(b)^{-1}, \eta\right]. \tag{2.3}$$

Then for $v_0 := v_{B_0, e_0}$ *we have* $B_0 \in \mathscr{B}^{v_0}$ *and* $(B_0 \backslash \{b\}) \cup \{e_0\} \in \mathscr{B}^{v_0}$ *for all* $b \in B_0$ *with* $(B_0 \backslash \{b\}) \cup \{e_0\} \in \mathscr{B}$. *Moreover, for any* $e \in E \backslash (B_0 \cup \{e_0\})$ *which is not a loop in* M *there exists some* $b \in B_0$ *with* $(B_0 \backslash \{b\}) \cup \{e\} \in \mathscr{B}^{v_0}$.

Furthermore, if $B_0 \cup \{e_0\}$ *is a circuit in* M, *then* $\mathscr{B}^{v_0}$ *is maximal among all* $\mathscr{B}^w$ *with* w *similar to* v.

Proof. Without loss of generality we may assume that M has no loops. Clearly, we have $v_0(B_0) = v(B_0)$, and for $b \in B_0$ with $(B_0 \backslash \{b\}) \cup \{e_0\} \in \mathscr{B}$ we obtain

$$\begin{aligned} v_0((B_0 \backslash \{b\}) \cup \{e_0\}) &= \eta(e_0) \cdot \eta(b)^{-1} \cdot v((B_0 \backslash \{b\}) \cup \{e_0\}) \\ &= v(B_0) = v_0(B_0). \end{aligned}$$

Furthermore, for $e \in E \backslash (B_0 \cup \{e_0\})$ and $b \in B_0$ with $(B_0 \backslash \{b\}) \cup \{e\} \in \mathscr{B}$ we obtain

$$\begin{aligned} v_0((B_0 \backslash \{b\}) \cup \{e\}) &= \eta(e) \cdot \eta(b)^{-1} \cdot v((B_0 \backslash \{b\}) \cup \{e\}) \\ &\leqslant v(B_0) = v_0(B_0) \end{aligned}$$

and $v_0((B_0 \backslash \{b\}) \cup \{e\}) = v_0(B_0)$ for at least one $b \in B_0$.

Thus we have $v_0(B) \leqslant v_0(B_0)$ for all $B \in \mathscr{B}$ with $\#(B_0 \backslash B) = m - 1$ and $v_0(B) = v_0(B_0)$ for all such B with the additional property $e_0 \in B$. So by Lemma 2.11 we have $B_0 \in \mathscr{B}^{v_0}$ and $(B_0 \backslash \{b\}) \cup \{e_0\} \in \mathscr{B}^{v_0}$ for all $b \in B_0$ with $(B_0 \backslash \{b\}) \cup \{e_0\} \in \mathscr{B}$. Moreover, for any $e \in E \backslash (B_0 \cup \{e_0\})$ there exists some $b \in B_0$ with $(B_0 \backslash \{b\}) \cup \{e\} \in \mathscr{B}^{v_0}$.

Finally, assume that $C := B \cup \{e_0\}$ is a circuit. Then by construction we have $\{e_0, b\} \in K_2(B_0, v_0)$ for all $b \in B_0$, and for every $e \in E \backslash C$ there exists some $b \in B_0$ with $\{e, b\} \in K_2(B_0, v_0)$. This means that the graph $G_2(B_0, v_0)$

is connected. The matroid M is also connected, because C is a circuit. Thus our last assertion follows from Proposition 2.14. ∎

COROLLARY 2.18. *Assume M has no loops and $v\colon E^m \to \bar{\Gamma}$ is a valuation of M. Assume $B_0 = \{e_1, ..., e_m\} \in \mathscr{B}$ and $e_0 \in E \setminus B_0$ are such that $C_0 := B_0 \cup \{e_0\}$ is a circuit in M. Furthermore, assume that $w\colon E^m \to \bar{\Gamma}$ is similar to v such that $C_0 \setminus \{e\} \in \mathscr{B}^w$ for all $e \in C_0$. Then the following four statements are equivalent:*

(i) *M^w has no loops.*

(ii) *$G_2(B_0, w)$ is connected.*

(iii) *$\mathscr{B}^w$ is maximal among all $\mathscr{B}^{w'}$ with w' similar to v.*

(iv) *w is equivalent to $v_0 := v_{B_0, e_0}$.*

Proof. (iv) $\Rightarrow$ (i) is a trivial consequence of Proposition 2.17. Since every $e \in B_0$ is connected to e_0 in $G_2(B_0, w)$, the implication (i) $\Rightarrow$ (ii) is also obvious.

(ii) $\Rightarrow$ (iii) follows from Proposition 2.14. Finally, (iii) implies (iv), since for any $\alpha \in \Gamma$ and $\eta'\colon E \to \Gamma$ with $v_0 = w[\alpha, \eta']$ one has $\eta'(e_i) = \eta'(e_0)$ for $1 \leqslant i \leqslant m$ in view of

$$\begin{aligned}
&\alpha \cdot \prod_{j=1}^{m} \eta'(e_j) \cdot w(B_0) \\
&\quad = v_0(B_0) = v_0((B_0 \setminus \{e_i\}) \cup \{e_0\}) \\
&\quad = \alpha \cdot \prod_{j=1}^{m} \eta'(e_j) \cdot \eta'(e_0) \cdot \eta'(e_i)^{-1} \cdot w((B_0 \setminus \{e_i\}) \cup \{e_0\})
\end{aligned}$$

and $w(B_0) = w((B_0 \setminus \{e_i\}) \cup \{e_0\})$. Therefore $(v_0)_{B_0}$ and w_{B_0} are necessarily equivalent. But $(v_0)_{B_0} = v_0$ and $w_{B_0} = w$ by Proposition 2.12, so v_0 and w must be equivalent. ∎

3. WEAK PREIMAGES OF VALUATED MATROIDS

In this section we prove the following simple but pleasing result which states that if M and M' are matroids on the same finite set, differing only by one base of M' which is not a base of M, then every valuation of M may be extended to M'.

PROPOSITION 3.1. *Assume E is some finite set and M, M' are matroids, both defined on E and of rank m, with $\mathscr{B}$ and $\mathscr{B}'$ as their sets of bases, respectively, such that $\mathscr{B}' = \mathscr{B} \cup \{B_0\}$ for some $B_0 \in \mathscr{B}' \setminus \mathscr{B}$. If $\Gamma = (\Gamma, \cdot, \leqslant)$*

is a linearly ordered abelian group and $v\colon E^m \to \bar{\Gamma}$ *is a valuation of* M, *then there exists some valuation* $v'\colon E^m \to \bar{\Gamma}$ *of* M' *such that* $v'(e_1, ..., e_m) = v(e_1, ..., e_m)$ *whenever* $\{e_1, ..., e_m\} \neq B_0$.

Proof. Choose some $\gamma \in \Gamma$ such that for all $\gamma_1, \gamma_2, \gamma_3 \in v(E^m) \cap \Gamma$ we have $\gamma \leqslant \gamma_1 \cdot \gamma_2 \cdot \gamma_3^{-1}$ and define $v'\colon E^m \to \bar{\Gamma}$ by

$$v'(e_1, ..., e_m) := \begin{cases} v(e_1, ..., e_m) & \text{if } \{e_1, ..., e_m\} \neq B_0 \\ \gamma & \text{otherwise.} \end{cases}$$

Clearly, v' satisfies (V0) and (V1). To verify (V2) we have only to show that for all $B \in \mathscr{B}$ and $e \in B_0 \backslash B$ there exists $f \in B \backslash B_0$ with

$$v'(B_0) \cdot v'(B) \leqslant v'((B_0 \backslash \{e\}) \cup \{f\}) \cdot v'((B \backslash \{f\}) \cup \{e\}) \tag{3.1}$$

and that, vice versa, for $f \in B \backslash B_0$ there exists $e \in B_0 \backslash B$ such that (3.1) holds also. But both facts follow from the strong exchange property for bases applied to the matroid M' by our choice of γ.

It is now trivial that v' is a valuation of M'. ∎

Remark. If $\mathscr{B} \subseteq \mathscr{B}'$ and $\#(\mathscr{B}' \backslash \mathscr{B}) \geqslant 2$, then there may exist valuations of M which do not extend to M'.

Look at the example of the projective plane M' over the field $\mathbb{F}_3$ with E as its set of points and $\mathscr{B}'$ as its set of bases. Consider some fixed line $L = \{e_1, e_2, e_3, e_4\}$ in M' and choose some fixed point $e_0 \in E \backslash L$. Moreover, put

$$\mathscr{B} := \{\{e_0, e_i, e_j\} \mid 1 \leqslant i < j \leqslant 4\}.$$

$\mathscr{B}$ is obviously the set of bases of some matroid M defined on E, and we have $\mathscr{B} \subseteq \mathscr{B}'$. The valuations of M correspond naturally and in a one-to-one fashion to the valuations of the restriction $M_0 := M \mid L = M' \mid L$. But M_0 is the uniform matroid of rank 2 with four elements. Thus Example 2.5 shows that M_0 and M are not rigid. However, we will show in Section 5 (see Theorem 5.11) that M' is rigid. Therefore Lemma 2.3 implies that those valuations of M which are not essentially trivial cannot be extended to M'.

We will now show that Proposition 3.1 does not hold for infinite E.

EXAMPLE 3.2. Assume $m \geqslant 3$, put $H := \{{}^t(x_1, ..., x_m) \in \mathbb{Q}^m \mid x_1 = 0\}$, let

$$f_i := {}^t(0, ..., 0, \underset{\uparrow}{1}, 0, ..., 0)$$

denote the ith unit vector in $\mathbb{R}^m$ for $1 \leqslant i \leqslant m$, put

$$e_0 := \sum_{i=2}^{m} f_i = {}^t(0, 1, ..., 1),\ B_0 := \{e_0\} \cup \{f_i \mid 2 \leqslant i \leqslant m\},$$

$$E := B_0 \cup (\mathbb{Q}^m \backslash H),$$

and let M denote the matroid defined on E by linear (in)dependence. Let $\mathscr{B} := \mathscr{B}_M$ denote the bases of M and put $\mathscr{B}' := \mathscr{B} \cup \{B_0\}$. Since B_0 is a circuit as well as a hyperplane in M, $\mathscr{B}'$ is the system of bases of some matroid M' defined on E. Choose some fixed prime number p and define $v\colon E^m \to \mathbb{Q}^+ \cup \{0\}$ by $v := v_p \circ \det$. Now assume there exists some valuation $v'\colon E^m \to \mathbb{Q}^+ \cup \{0\}$ of M' such that $v'(e_1, ..., e_m) = v(e_1, ..., e_m)$ whenever $\{e_1, ..., e_m\} \neq B_0$ and put $\gamma := v'(e_0, f_2, ..., f_m)$. Necessarily, we have $\gamma = v'(B_0) > 0$.

We will now show that there exists some $B \in \mathscr{B}_M$ with $e_0 \notin B$ such that for all $e \in B \backslash B_0$ we have

$$v'(B_0) \cdot v'(B) > v'((B_0 \backslash \{e_0\}) \cup \{e\}) \cdot v'((B \backslash \{e\}) \cup \{e_0\}). \tag{3.2}$$

Choose some $n \in \mathbb{Z}$ with $p^{-n} < \gamma$. Put $e_1 := {}^t(1, 0, ..., 0)$, $e_2 := {}^t(1, p^{-n}, 0, ..., 0)$ and $e_i := f_i$ for $3 \leqslant i \leqslant m$. Then we have $B := \{e_1, ..., e_m\} \subseteq E$ and

$$v'(B_0) \cdot v'(B) = \gamma \cdot p^n > 1,$$

$$v'((B_0 \backslash \{e_0\}) \cup \{e_i\}) \cdot v'((B \backslash \{e_i\}) \cup \{e_0\}) = 1 \cdot 1 = 1 \qquad \text{for} \quad i \in \{1, 2\}.$$

Thus (3.2) holds for all $e \in B \backslash B_0$.

4. Valuated Matroids and Matroids with Coefficients

Fuzzy rings and matroids with coefficients in a fuzzy ring have been introduced in [D1].

In this section we will show that valuated matroids are nothing but matroids with coefficients in an appropriate fuzzy ring. To this end we recall the definition of a fuzzy ring.

Definition 4.1. A fuzzy ring $K = (K; +; \cdot; \varepsilon; K_0)$ consists of a set K together with two compositions

$$+ : K \times K \to K\colon (\kappa, \lambda) \mapsto \kappa + \lambda$$

and

$$\cdot : K \times K \to K\colon (\kappa, \lambda) \mapsto \kappa \cdot \lambda,$$

a specified element $\varepsilon \in K$ and a specified subset $K_0 \subseteq K$ such that the following holds:

(FR0) $(K, +)$ and $(K, \cdot)$ are abelian semigroups with neutral elements 0 and 1, respectively;

(FR1) $0 \cdot \kappa = 0$ for all $\kappa \in K$;

(FR2) $\alpha \cdot (\kappa_1 + \kappa_2) = \alpha \cdot \kappa_1 + \alpha \cdot \kappa_2$ for all $\kappa_1, \kappa_2 \in K$ and $\alpha \in K^* := \{\beta \in K \mid 1 \in \beta \cdot K\}$, the group of *units* in K;

(FR3) $\varepsilon^2 = 1$;

(FR4) $K_0 + K_0 \subseteq K_0$, $K \cdot K_0 \subseteq K_0$, $0 \in K_0$, $1 \notin K_0$;

(FR5) for $\alpha \in K^*$ one has $1 + \alpha \in K_0$ if and only if $\alpha = \varepsilon$;

(FR6) $\kappa_1, \kappa_2, \lambda_1, \lambda_2 \in K$ and $\kappa_1 + \lambda_1$, $\kappa_2 + \lambda_2 \in K_0$ implies $\kappa_1 \cdot \kappa_2 + \varepsilon \cdot \lambda_1 \cdot \lambda_2 \in K_0$;

(FR7) $\kappa, \lambda, \kappa_1, \kappa_2 \in K$ and $\kappa + \lambda \cdot (\kappa_1 + \kappa_2) \in K_0$ implies $\kappa + \lambda \cdot \kappa_1 + \lambda \cdot \kappa_2 \in K_0$.

Examples. (i) The commutative rings $R = (R; +; \cdot)$ with $1 \in R$ are (in a canonical correspondence to) exactly those fuzzy rings $(K; +; \cdot; \varepsilon; K_0)$ for which $K_0 = \{0\}$. In this case we have necessarily $\varepsilon = -1$.

(ii) If $K = (K; +; \cdot; \varepsilon; K_0)$ is a fuzzy ring and if $U \leqslant K^*$ is a subgroup of the group of units, then we can form the "quotient fuzzy ring"

$$K/U := (\mathscr{P}(K)^U; \dot{+}; \cdot; \varepsilon \cdot U; \mathscr{P}(K)_0^U),$$

where $\mathscr{P}(K)^U$ denotes the U-invariant subsets of K (i.e., $F \in \mathscr{P}(K)^U$ if and only if $U \cdot F = F$), which are added and multiplied as "complexes"; that is by

$$T_1 \dot{+} T_2 := \{\kappa_1 \dot{+} \kappa_2 \mid \kappa_1 \in T_1, \kappa_2 \in T_2\} \qquad (T_1, T_2 \in \mathscr{P}(K)^U),$$

and where $\mathscr{P}(K)_0^U$ denotes those U-invariant subsets $T \subseteq K$ for which $T \cap K_0 \neq \varnothing$.

Note that $T \in K/U$ is a unit if and only if $T = \alpha \cdot U$ for some $\alpha \in K^*$. In [DW3, (4.7)] it is shown that $\mathbb{R}/\mathbb{R}^*$ is an appropriate domain of coefficients for combinatorial geometries or matroids in the ordinary sense.

(iii) If K is a fuzzy ring and if $L \subseteq K$ is the smallest subset of K with $K^* \cup \{0\} \subseteq L$ and $L \dot{+} L \subseteq L$, then $(L; +; \cdot; \varepsilon; L \cap K_0)$ is also a fuzzy ring. This is particularly interesting in case K is of the form K_1/U_1 for some fuzzy ring K_1 and some subgroup $U_1 \leqslant K_1^*$, in which case we write $K_1//U_1$ for L. By [DW2, §6], the fuzzy rings $\mathbb{R}/\mathbb{R}^+$ and $\mathbb{R}//\mathbb{R}^+$ are appropriate domains of coefficients for oriented matroids.

These and some further examples of fuzzy rings are considered in detail in [D1, (1.3)].

In view of [DW2, Sect. 4] we will now give the definition of matroids with coefficients of finite rank in terms of Grassmann–Plücker maps; this definition is the most convenient one for our purposes.

DEFINITION 4.2. Assume E is some set and $m \in \mathbb{N}$. *A matroid M of rank m, defined on E, with coefficients in a fuzzy ring* $K = (K; +; \cdot; \varepsilon; K_0)$ is an equivalence class of maps $b: E^m \to K^* \cup \{0\}$ satisfying the following conditions:

(GP0) There exist $e_1, ..., e_m \in E$ with $b(e_1, ..., e_m) \neq 0$;

(GP1) b is ε-alternating; this means, for $e_1, ..., e_m \in E$ and every odd permutation $\tau \in \Sigma_m$ we have

$$b(e_{\tau(1)}, ..., e_{\tau(m)}) = \varepsilon \cdot b(e_1, ..., e_m)$$

and in case $\# \{e_1, ..., e_m\} < m$ we have $b(e_1, ..., e_m) = 0$;

(GP2) for all $e_0, ..., e_m, f_2, ..., f_m \in E$ we have

$$\sum_{i=0}^{m} \varepsilon^i \cdot b(e_0, ..., \hat{e}_i, ..., e_m) \cdot b(e_i, f_2, ..., f_m) \in K_0, \tag{4.1}$$

where two such maps b, b' are called equivalent if $b \equiv u \cdot b'$ for some $u \in K^*$.

If $b: E^m \to K^* \cup \{0\}$ satisfies (GP0), (GP1), and (GP2), then b is called a *Grassmann–Plücker map of degree m* with values in K, defined on E^m.

Remarks. (i)If M and b are as above, then we write also $M = M_b$.

(ii) The relations (4.1) are called *Grassmann–Plücker relations.*

(iii) According to [DW2, Proposition 4.1] the set $\mathscr{B} = \mathscr{B}_M$ of bases of $M = M_b$, given by

$$\mathscr{B} := \left\{ \{e_1, ..., e_m\} \in \binom{E}{m} \,\middle|\, b(e_1, ..., e_m) \neq 0 \right\},$$

forms the set of bases of a combinatorial geometry (or matroid) $\underline{M} = \underline{M_b}$ in the usual sense.

(iv) If K, E, and b are as above and $U \leqslant K^*$ is a subgroup of K^*, then $b_U: E^m \to (K^*/U) \cup \{0\}: (e_1, ..., e_m) \mapsto b(e_1, ..., e_m) \cdot U$ is a Grassmann–Plücker map of degree m with values in K/U or, equivalently, in $K/\!/U$.

EXAMPLE. Assume $K = \mathbb{F}$ is a field and $E \subseteq \mathbb{F}^m$ is a spanning subset of the vector space $\mathbb{F}^m$. Then $b: E^m \to \mathbb{F}$ defined by

$$b(e_1, ..., e_m) := \det(e_1, ..., e_m)$$

is a Grassmann–Plücker map. In this case the relations (4.1) state the well-known identity of Grassmann:

For all $e_0, ..., e_m, f_2, ..., f_m \in E$ we have

$$\sum_{i=0}^{m} (-1)^i \cdot \det(e_0, ..., \hat{e}_i, ..., e_m) \cdot \det(e_i, f_2, ..., f_m) = 0. \tag{4.1a}$$

(See also (1.3) in case $\mathbb{F} = \mathbb{Q}$.)

$\underline{M_b}$ is just the matroid defined on E by linear (in)dependence.

In the sequel we assume that $\Gamma = (\Gamma, \cdot, \leqslant)$ is a linearly ordered abelian group. We construct a fuzzy ring $K = K_\Gamma$ which may be interpreted as the domain of coefficients for valuated matroids with values in Γ.

To this end we define for each $\gamma \in \Gamma$ the subset

$$\tilde{\gamma} := \{\gamma' \in \Gamma \mid \gamma' \leqslant \gamma\},$$

we put

$$K_\Gamma := \{\Delta \subseteq \Gamma \mid \# \Delta = 1 \text{ or } \Delta = \tilde{\gamma} \text{ for some } \gamma \in \Gamma\} \subseteq \mathscr{P}(\Gamma)$$

and

$$\tilde{\Gamma} := \{\tilde{\gamma} \mid \gamma \in \Gamma\} \subseteq K_\Gamma,$$

and we identify each $\gamma \in \Gamma$ with the one-element subset $\{\gamma\} \in K_\Gamma$ so that

$$K_\Gamma = \Gamma \mathbin{\dot\cup} \tilde{\Gamma}.$$

Finally, for all $\Delta, \Delta' \subseteq \Gamma$ we define

$$\Delta \cdot \Delta' := \{\delta \cdot \delta' \mid \delta \in \Delta, \delta' \in \Delta'\},$$

$$\Delta \ominus \Delta' := \{\delta \in \Delta \mid \text{there exists } \delta' \in \Delta' \text{ with } \delta' < \delta\},$$

$$\Delta + \Delta' := (\Delta \ominus \Delta') \cup (\Delta' \ominus \Delta) \cup \bigcup_{\delta \in \Delta \cap \Delta'} \tilde{\delta},$$

and

$$\Delta +_2 \Delta' := (\Delta \ominus \Delta') \cup (\Delta' \ominus \Delta) \cup \bigcup_{\delta \in \Delta \cap \Delta'} \tilde{\delta} \setminus (\{\delta\} \cap \Gamma).$$

Note that $\Delta +_2 \Delta' = \Delta + \Delta'$ unless there exists $\delta \in \Delta \cap \Delta' \cap \Gamma$ with $\delta = \min \Delta = \min \Delta'$ in which case $(\Delta + \Delta') \setminus (\Delta +_2 \Delta') = \{\delta\}$.

Furthermore, note that $K_\Gamma \cdot K_\Gamma \subseteq K_\Gamma$ and $K_\Gamma + K_\Gamma \subseteq K_\Gamma$ (but not necessarily $K_\Gamma +_2 K_\Gamma \subseteq K_\Gamma$) and that for the elements 0 and $\alpha, \beta \in \Gamma$ we have the following addition and multiplication table in K_Γ:

$+$	0	α	β	$\tilde{\alpha}$	$\tilde{\beta}$
0	0	α	β	$\tilde{\alpha}$	$\tilde{\beta}$
α	α	$\tilde{\alpha}$	β	$\tilde{\alpha}$	$\tilde{\beta}$
β	β	β	$\tilde{\beta}$	β	$\tilde{\beta}$
$\tilde{\alpha}$	$\tilde{\alpha}$	$\tilde{\alpha}$	β	$\tilde{\alpha}$	$\tilde{\beta}$
$\tilde{\beta}$	$\tilde{\beta}$	$\tilde{\beta}$	$\tilde{\beta}$	$\tilde{\beta}$	$\tilde{\beta}$

$(\alpha < \beta)$

$\cdot$	0	β	$\tilde{\beta}$
0	0	0	0
α	0	$\alpha \cdot \beta$	$\widetilde{\alpha \cdot \beta}$
$\tilde{\alpha}$	0	$\widetilde{\alpha \cdot \beta}$	$\widetilde{\alpha \cdot \beta}$

We claim:

THEOREM 4.3. *With this addition and multiplication* $(K_\Gamma; +; \cdot; 1; \{0\} \cup \tilde{\Gamma})$ *is a fuzzy ring with* $K_\Gamma^* = \Gamma$ *such that for any set* E *and every* $m \in \mathbb{N}$ *a map* $v\colon E^m \to \bar{\Gamma} = K_\Gamma^* \cup \{0\}$ *defines a valuated matroid of rank* m *on* E *if and only if it is a Grassmann–Plücker map of degree* m *with values in* K_Γ.

Proof. Though we could give a direct computational proof of this theorem, we prefer to derive it from several lemmata which will also help to clarify the motivation for our definition of K_Γ.

LEMMA 4.4. *For any linearly ordered abelian group* $\Gamma = (\Gamma, \cdot, \leqslant)$ *there exists a field* $\mathbb{F}$ *together with a surjective non-archimedian valuation* $\Phi\colon \mathbb{F} \twoheadrightarrow \bar{\Gamma}$, *i.e., a surjective map* $\Phi\colon \mathbb{F} \to \bar{\Gamma}$ *such that for all* $x, y \in \mathbb{F}$ *one has*

$$\Phi(x) = 0 \Leftrightarrow x = 0, \tag{4.2a}$$

$$\Phi(x+y) \leqslant \max(\Phi(x), \Phi(y)), \tag{4.2b}$$

$$\Phi(x \cdot y) = \Phi(x) \cdot \Phi(y) \tag{4.2c}$$

and therefore also

$$\Phi(x) = \Phi(-x), \tag{4.2d}$$

$$\Phi(x+y) = \Phi(x) \qquad \textit{if} \quad \Phi(y) < \Phi(x). \tag{4.2e}$$

Proof. This is well known (cf. [S, Chap. I, Sect. 6]). One way to construct some such $\mathbb{F}$ is the following one. For an arbitrary field $\mathbb{F}_0$ consider the group ring

$$R_0 := \mathbb{F}_0[\Gamma] = \left\{ \sum_{\gamma \in \Gamma} a_\gamma \gamma \mid a_\gamma \in \mathbb{F}_0,\ a_\gamma = 0 \text{ for almost all } \gamma \in \Gamma \right\}$$

together with the map

$$\Phi_0 : R_0 \to \bar{\Gamma} : \sum a_\gamma \gamma \mapsto \min(\gamma \mid a_\gamma \neq 0)^{-1}$$

(using the convention $\min(\emptyset)^{-1} := 0$). Obviously, $\Phi_0(x) = 0$ if and only if $x = 0$, $\Phi_0(x + y) \leqslant \max(\Phi_0(x), \Phi_0(y))$, and $\Phi_0(x \cdot y) = \Phi_0(x) \cdot \Phi_0(y)$ for all $x, y \in R_0$, in particular $x \cdot y \neq 0$ if $x \neq 0 \neq y$. Hence R_0 is an integral domain and we may choose $\mathbb{F}$ to be the quotient field $\mathbb{F}'$ of R_0 and Φ to be the (well-defined!) map

$$\Phi_1 : \mathbb{F}' \to \bar{\Gamma} : x/y \mapsto \Phi_0(x)/\Phi_0(y) \qquad (x, y \in R_0;\ y \neq 0).$$

Another possibility is to define $\mathbb{F}$ as the field $\mathbb{F}''$ of all Γ-valued formal power series with coefficients from $\mathbb{F}_0$, that is of all formal sums $\sum_{\gamma \in \Gamma} a_\gamma \gamma$ ($a_\gamma \in \mathbb{F}_0$) for which $\{\gamma \mid a_\gamma \neq 0\}$ is a well ordered subset of Γ, with (well-defined!) sum $\sum a_\gamma \gamma + \sum b_\gamma \gamma := \sum (a_\gamma + b_\gamma)\gamma$ and product $\sum a_\gamma \gamma \cdot \sum b_\gamma \gamma := \sum (\sum_{\gamma_1 \cdot \gamma_2 = \gamma} a_{\gamma_1} b_{\gamma_2})\gamma$ and to define Φ as the map

$$\Phi_2 : \mathbb{F}'' \to \bar{\Gamma} : \sum a_\gamma \gamma \mapsto \begin{cases} 0 & \text{if } \ a_\gamma = 0 \text{ for all } \gamma \\ \min(\gamma \mid a_\gamma \neq 0)^{-1} & \text{otherwise.} \end{cases}$$

In a way, $\mathbb{F}''$ can be viewed as the *completion* of $\mathbb{F}'$ relative to the valuation Φ_1. ∎

Next assume that Γ is a linearly ordered abelian group, $\mathbb{F}$ is a field and $\Phi : \mathbb{F} \twoheadrightarrow \bar{\Gamma}$ is a surjective non-archimedian valuation. Note that $R_\Phi := \Phi^{-1}(\bar{1}) = \{x \in \mathbb{F} \mid \Phi(x) \leqslant 1\}$ is a subring of $\mathbb{F}$ with a unique maximal ideal $\mathbf{m}_\Phi := \Phi^{-1}(\bar{1} \setminus \{1\}) = \{x \in \mathbb{F} \mid \Phi(x) < 1\}$ and with $U_\Phi := \Phi^{-1}(1) = R_\Phi \setminus \mathbf{m}_\Phi$ as its group of units. Let $\mathbb{F}_\Phi := R_\Phi / \mathbf{m}_\Phi$ denote its residue class field and note that for the examples $\mathbb{F} = \mathbb{F}', \mathbb{F}''$, constructed in the proof of Lemma 4.4, we have $\mathbb{F}_\Phi = \mathbb{F}_0$. Note also that Φ induces a canonical one-to-one correspondence between $\mathscr{P}(\mathbb{F})^{U_\Phi}$, the set of U_Φ-invariant subsets of $\mathbb{F}$, and the set $\mathscr{P}(\bar{\Gamma})$, given by

$$\Delta \mapsto \Delta_{\mathbb{F}} := \Phi^{-1}(\Delta) \qquad (\Delta \subseteq \bar{\Gamma}).$$

We claim

LEMMA 4.5. *For all $\Delta, \Delta' \subseteq \bar{\Gamma}$ we have*

$$\Delta_{\mathbb{F}} \cdot \Delta'_{\mathbb{F}} = (\Delta \cdot \Delta')_{\mathbb{F}}$$

and

$$\Delta_{\mathbb{F}} + \Delta'_{\mathbb{F}} = \begin{cases} (\Delta +_2 \Delta')_{\mathbb{F}} & \text{if } \ \#\mathbb{F}_\Phi = 2 \\ (\Delta + \Delta')_{\mathbb{F}} & \text{otherwise.} \end{cases}$$

Proof. Obviously $\Delta_{\mathbb{F}} \cdot \Delta'_{\mathbb{F}} \subseteq (\Delta \cdot \Delta')_{\mathbb{F}}$ in view of (4.2c). Vice versa, if $x \in (\Delta \cdot \Delta')_{\mathbb{F}}$, say $\Phi(x) = \delta \cdot \delta'$ with $\delta \in \Delta$ and $\delta' \in \Delta'$, we may choose $y \in \delta_{\mathbb{F}} \subseteq \Delta_{\mathbb{F}}$ arbitrarily, and in case $y \neq 0$ we may put $y' := y^{-1} \cdot x \in \delta'_{\mathbb{F}} \subseteq \Delta'_{\mathbb{F}}$ to find that $x = y \cdot y' \in \Delta_{\mathbb{F}} \cdot \Delta'_{\mathbb{F}}$, while in case $y = 0$ we may choose $y' \in \delta'_{\mathbb{F}}$ arbitrarily and still have $x = 0 = y \cdot y' \in \Delta_{\mathbb{F}} \cdot \Delta'_{\mathbb{F}}$, since $\Phi(x) = \Phi(y) \cdot \Phi(y') = 0$.

It is also an obvious consequence of (4.2b) and (4.2e) that $\Delta_{\mathbb{F}} + \Delta'_{\mathbb{F}} \subseteq (\Delta + \Delta')_{\mathbb{F}}$ and that $\Delta_{\mathbb{F}} + \Delta'_{\mathbb{F}} \subseteq (\Delta_2 + \Delta')_{\mathbb{F}}$ in case $\# \mathbb{F}_{\Phi} = 2$, that is $U_{\Phi} + U_{\Phi} = \mathbf{m}_{\Phi}$ and therefore $\delta_{\mathbb{F}} + \delta_{\mathbb{F}} \subseteq (\tilde{\delta} \setminus \{\delta\})_{\mathbb{F}}$ for all $\delta \in \Gamma$.

Vice versa, assume that $x \in (\Delta + \Delta')_{\mathbb{F}}$. If $\Phi(x) = \delta \in \Delta$ and there exists $\delta' \in \Delta'$ with $\delta' < \delta$, then choose $y \in \delta'_{\mathbb{F}}$ and observe that $x = (x - y) + y \in \Delta_{\mathbb{F}} + \Delta'_{\mathbb{F}}$ in view of $\Phi(x - y) = \Phi(x) = \delta \in \Delta$ because of (4.2e) and $\Phi(-y) = \Phi(y) = \delta' < \delta = \Phi(x)$. Similarly $x \in \Delta_{\mathbb{F}} + \Delta'_{\mathbb{F}}$ if $\Phi(x) = \delta' \in \Delta'$ and there exists $\delta \in \Delta$ with $\delta < \delta'$. Finally, if $\Phi(x) \leqslant \delta \in \Delta \cap \Delta'$, then in case $\Phi(x) < \delta$ we may choose $y \in \delta_{\mathbb{F}}$ arbitrarily to derive $x = y + (x - y) \in \delta_{\mathbb{F}} + \delta_{\mathbb{F}} \subseteq \Delta_{\mathbb{F}} + \Delta'_{\mathbb{F}}$ in view of $\Phi(x - y) = \Phi(y)$ because of $\Phi(x) < \Phi(y) = \Phi(-y)$, while in case $\Phi(x) = \delta$ and $\# \mathbb{F}_{\Phi} \neq 2$, that is $U_{\Phi} + U_{\Phi} \not\subseteq \mathbf{m}_{\Phi}$, we may choose $y_1, y_2 \in U_{\Phi}$ with $y := y_1 + y_2 \in U_{\Phi}$ to derive $x = y_1 \cdot x \cdot y^{-1} + y_2 \cdot x \cdot y^{-1} \in \delta_{\mathbb{F}} + \delta_{\mathbb{F}} \subseteq \Delta_{\mathbb{F}} + \Delta'_{\mathbb{F}}$. ∎

It follows immediately from the last lemma that with Γ, $\mathbb{F}$, and Φ as above K_Γ coincides with the fuzzy ring $\mathbb{F}//U_{\Phi}$ in case $\# \mathbb{F}_{\Phi} \neq 2$, and hence it follows from the last two lemmata that K_Γ is always a fuzzy ring. Similarly, in case $\# \mathbb{F}_{\Phi} = 2$ the fuzzy ring $\mathbb{F}//U_{\Phi}$ can be identified with

$$K_\Gamma^{(2)} := \{\Delta \subseteq \bar{\Gamma} \mid \# \Delta = 1 \text{ or } \Delta = \tilde{\gamma}^{(2)} := \tilde{\gamma} \setminus \{\gamma\} \text{ for some } \gamma \in \Gamma\} \subseteq \mathscr{P}(\bar{\Gamma})$$

or, more precisely, with $(K_\Gamma^{(2)}; +_2; \cdot; 1; \{\Delta \in K_\Gamma^{(2)} \mid 0 \in \Delta\})$. Matroids with coefficients in this fuzzy ring will be studied elsewhere.

It remains to show that valuated matroids $(E, v\colon E^m \to \bar{\Gamma})$ of rank m, defined on a set E, are nothing but Grassmann–Plücker maps of degree m, defined on E^m, with values in K_Γ. Indeed, the conditions (V0) and (GP0) coincide as well as the conditions (V1) and (GP1), taking into account that $\varepsilon = 1$ in K_Γ. To show that also (V2) and (GP2) are equivalent we use the obvious

Lemma 4.6. *If $\gamma_0, \gamma_1, ..., \gamma_m \in \bar{\Gamma}$ and if $\gamma := \max(\gamma_0, \gamma_1, ..., \gamma_m)$, then in K_Γ we have*

$$\gamma_0 + \cdots + \gamma_m = \begin{cases} \gamma & \text{if } \# \{i \mid \gamma_i = \gamma\} = 1 \text{ or } \gamma_i = 0 \text{ for } i = 0, ..., m \\ \tilde{\gamma} & \text{otherwise.} \end{cases}$$

Now assume $e_0, ..., e_m, f_2, ..., f_m \in E$. To show that (V2) implies (GP2) put

$$\gamma := \max(v(e_0, ..., \hat{e}_i, ..., e_m) \cdot v(e_i, f_2, ..., f_m),\ i = 0, ..., m),$$

say $\gamma = v(e_1, ..., e_m) \cdot v(e_0, f_2, ..., f_m)$. By (V2) there exists some $i \in \{1, ..., m\}$ with

$$\begin{aligned}\gamma &= v(e_1, ..., e_m) \cdot v(e_0, f_2, ..., f_m) \\ &\leqslant v(e_0, ..., \hat{e}_i, ..., e_m) \cdot v(e_i, f_2, ..., f_m) \leqslant \gamma\end{aligned}$$

and therefore

$$v(e_0, ..., \hat{e}_i, ..., e_m) \cdot v(e_i, f_2, ..., f_m) = \gamma,$$

that is

$$\#\{i \mid v(e_0, ..., \hat{e}_i, ..., e_m) \cdot v(e_i, f_2, ..., f_m) = \gamma\} > 1$$

and therefore

$$0 \in \sum_{i=0}^{m} v(e_0, ..., \hat{e}_i, ..., e_m) \cdot v(e_i, f_2, ..., f_m)$$

by Lemma 4.6.

Vice versa, if (GP2) holds and we have $e_0, ..., e_m, f_2, ..., f_m \in E$, then

$$0 \in \sum_{i=0}^{m} v(e_0, ..., \hat{e}_i, ..., e_m) \cdot v(e_i, f_2, ..., f_m)$$

together with Lemma 4.6 implies that

$$v(e_1, ..., e_m) \cdot v(e_0, f_2, ..., f_m) > v(e_0, ..., \hat{e}_i, ..., e_m) \cdot v(e_i, f_2, ..., f_m)$$

cannot hold for all $i = 1, ..., m$.

Finally, we have to remark that two Grassmann–Plücker maps $v, v'\colon E^m \to \bar{\Gamma}$ are equivalent if and only if v and v' are equivalent valuations if and only if $v \equiv \gamma \cdot v'$ for some $\gamma \in \Gamma$. ∎

We will now show

PROPOSITION 4.7. *The fuzzy ring K_Γ is distributive; that is, for all $\lambda, \gamma, \delta \in K_\Gamma$ we have*

$$\lambda \cdot (\gamma + \delta) = \lambda \cdot \gamma + \lambda \cdot \delta. \tag{4.3}$$

Proof. Put $K := K_\Gamma$ and $\tilde{\kappa} := \kappa$ for all $\kappa \in K_0$. Then by the addition and multiplication table in K we have $\widetilde{\gamma + \delta} = \tilde{\gamma} + \tilde{\delta}$ and $\widetilde{\gamma \cdot \delta} = \tilde{\gamma} \cdot \tilde{\delta} = \tilde{\gamma} \cdot \delta$ for all $\gamma, \delta \in K$.

To prove (4.3) we may assume $\lambda = \tilde{\kappa} \in \tilde{\Gamma}$ for some $\kappa \in \Gamma$, because otherwise we are done by (FR1) or (FR2). Then we obtain by (FR2) and our addition and multiplication table

$$\begin{aligned} \lambda \cdot (\gamma + \delta) &= \tilde{1} \cdot \kappa \cdot (\gamma + \delta) = \tilde{1} \cdot (\kappa \cdot \gamma + \kappa \cdot \delta) \\ &= 1 \cdot (\tilde{\kappa} \cdot \tilde{\gamma} + \tilde{\kappa} \cdot \tilde{\delta}) = \lambda \cdot \gamma + \lambda \cdot \delta. \quad \blacksquare \end{aligned}$$

Theorem 4.3 together with [DW2, Theorem 4.4] implies that valuated matroids of finite rank, defined on a set E and with values in K_Γ, can also be defined in terms of subsets $\mathscr{R} \subseteq K_\Gamma^E$ of maps from E into $K = K_\Gamma$, satisfying the condition (M), stated in [D1, Sect. 3.5]. In detail, we define for any map $r: E \to K$ its *support* $\underline{r} := \{e \in E \mid r(e) \neq 0\}$ and its *essential support* $\underline{\underline{r}} := \{e \in E \mid r(e) \notin K_0\} = \{e \in E \mid r(e) \in \Gamma = K^* = K \backslash K_0\}$, for any two maps $r, s: E \to K$ with $\#(\underline{r} \cap \underline{s}) < \infty$ we put $\langle r \mid s \rangle := \sum_{e \in E} r(e) \cdot s(e)$, and for any $e \in E$ we define a map

$$r \wedge_e s: E \to K: f \mapsto \begin{cases} 0 & \text{if } f = e \\ s(e) r(f) + r(e) s(f) & \text{if } f \neq e. \end{cases}$$

Finally, for any $\mathscr{R} \subseteq K^E$ we define

$$[\mathscr{R}] = \{(\cdots((r_0 \wedge_{e_1} r_1) \wedge_{e_2} r_2) \cdots \wedge_{e_n} r_n) \mid r_0, r_1, ..., r_n \in \mathscr{R}; e_1, ..., e_n \in E\},$$

and for any $\mathscr{R}, \mathscr{R}' \subseteq K^E$ we write $\mathscr{R} \leqslant \mathscr{R}'$ if for every $r' \in \mathscr{R}'$ and $e \in \underline{\underline{r}}'$ there exists some $r \in \mathscr{R}$ with $e \in \underline{r} = \underline{\underline{r}} \subseteq \underline{r}'$. Then a set $\mathscr{R} \subseteq K^E$ of maps from E into K is said to satisfy condition (M) if $\mathscr{R} \leqslant [\mathscr{R}]$, two such sets $\mathscr{R}, \mathscr{R}' \subseteq K^E$ are defined to be $\sim^M$-equivalent if $\mathscr{R} \leqslant \mathscr{R}'$ and $\mathscr{R}' \leqslant \mathscr{R}$, and a matroid, defined on E and with coefficients in K, is defined to be an $\sim^M$-equivalence class of such sets $\mathscr{R}, \mathscr{R}', ... \subseteq K^E$, satisfying condition (M). This definition is justified since it has been shown in [DW2, Sect. 4] that for every Grassmann–Plücker map or valuation $v: E^m \to \bar{\Gamma}$ the set

$$\begin{aligned} \mathscr{R}_v := \{r: E \to \bar{\Gamma} \mid & \text{there exist } e_0, ..., e_m \in E \text{ and } \gamma \in \Gamma \text{ with} \\ & r(e) = \gamma \cdot v(e_0, ..., \hat{e}_i, ..., e_m) \text{ if } e = e_i \text{ and } r(e) = 0 \text{ if} \\ & e \notin \{e_0, ..., e_m\}\} \end{aligned}$$

satisfies condition (M), for two Grassmann–Plücker maps $v: E^m \to \bar{\Gamma}$ and $v': E^{m'} \to \bar{\Gamma}$ one has $\mathscr{R}_v \sim^M \mathscr{R}_{v'}$ if and only if $\mathscr{R}_v = \mathscr{R}_{v'}$ if and only if $m = m'$ and $v = \gamma \cdot v'$ for some $\gamma \in \Gamma$, and for every $\mathscr{R} \subseteq K^E$, satisfying condition (M), there exists some valuation v with $\mathscr{R} \sim^M \mathscr{R}_v$ in which case one has $\mathscr{R}_v = \{\gamma \cdot r \mid \gamma \in \Gamma,\ r \in \mathscr{R},\ \underline{r} = \underline{\underline{r}} \neq \varnothing$, and $\underline{r}' \subsetneqq \underline{r}$ for some $r' \in \mathscr{R}$ only if $\underline{\underline{r}}' = \varnothing\}$.

Moreover, it was shown that for v^* as defined in Proposition 1.4 the set $\mathscr{R}_{v^*}$ coincides with

$$\{s\colon E \to \bar{\Gamma} \mid \text{there exist } \gamma \in \Gamma \text{ and } e_2, ..., e_m \in E \text{ with } s(e) = \gamma \cdot v(e, e_2, ..., e_m) \text{ for all } e \in E\},$$

and $\mathscr{R}^v := \bigcup_{\mathscr{R} \sim^M \mathscr{R}_v} \mathscr{R}$ coincides with $\{r\colon E \to K \mid \langle r \mid s\rangle \in K_0 \text{ for all } s \in \mathscr{R}_{v^*}\}$, satisfies also condition (M) and is $\sim^M$-equivalent to $\mathscr{R}_v$. For obvious reasons $\mathscr{R}_v$ and $\mathscr{R}^v$ are called the minimal and the maximal presentation of the matroid M_v, defined by v, respectively.

In addition (cf. [DW6]), Proposition 4.7 and the fact that $K = K^* \cup K_0$ for $K = K_\Gamma$ together imply that K_Γ is a *perfect fuzzy ring*; that is, for every valuation v one has $\langle r \mid s\rangle \in K_0$ for all $r \in \mathscr{R}^v$ and $s \in \mathscr{R}^{v^*}$.

As a further consequence of these results we will establish a connection between the results of Section 2 and Klingenberg's projective homomorphisms. Observe that—generalizing the valuations v_p of $\mathbb{Q}$—for any linearly ordered abelian group Γ and for any field $\mathbb{F}$ with a (not even necessarily surjective) non-archimedean valuation $\Phi\colon \mathbb{F} \to \bar{\Gamma}$ any *representation* $g\colon E \to \mathbb{F}^m$ of a combinatorial geometry M of rank m defined on a set E (that is any map $g\colon E \to \mathbb{F}^m$ with $\det(g(e_1), ..., g(e_m)) \neq 0$ if and only if $\{e_1, ..., e_m\} \in \mathscr{B}_M$ for all $e_1, ..., e_m \in E$) induces a valuation $v_\Phi\colon E^m \to \bar{\Gamma}$: $(e_1, ..., e_m) \mapsto \Phi(\det(g(e_1), ..., g(e_m)))$. If a given valuation v of M is equivalent to some such v_Φ, we say that the valuated matroid (E, v) is representable over $\mathbb{F}$ (relative to Φ).

Let $\mathscr{P}_{m-1}(\mathbb{F})$ denote the $(m-1)$-dimensional projective space over $\mathbb{F}$. Of course, $\mathscr{P}_{m-1}(\mathbb{F})$ is a matroid of rank m in a canonical sense: $\{\mathbb{F} \cdot x_1, ..., \mathbb{F} \cdot x_m\}$ is a base of $\mathscr{P}_{m-1}(\mathbb{F})$ if and only if $\{x_1, ..., x_m\}$ is a base of the vector space $\mathbb{F}^m$.

If $\Phi\colon \mathbb{F} \to \bar{\Gamma}$ is as above, then a corresponding valuation $v_\Phi\colon (\mathscr{P}_{m-1}(\mathbb{F}))^m \to \bar{\Gamma}$ of $\mathscr{P}_{m-1}(\mathbb{F})$ is given by

$$v_\Phi(e_1, ..., e_m) := \Phi(\det(x_1, ..., x_m)) \cdot \prod_{i=1}^{m} (\max(\Phi(x_{i1}), ..., \Phi(x_{im})))^{-1},$$

where $e_i = \mathbb{F} \cdot x_i$ and $x_i = {}^t(x_{i1}, ..., x_{im})$ for $1 \leqslant i \leqslant m$.

Put

$$C := \left\{ \mathbb{F} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbb{F} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, ..., \mathbb{F} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \mathbb{F} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \right\}.$$

C is a circuit in $\mathscr{P}_{m-1}(\mathbb{F})$ with $v_\Phi(C\setminus\{e\})=1$ for all $e\in C$. Since by definition $v_\Phi(e_1, ..., e_m)\leqslant 1$ for all $e_1, ..., e_m\in\mathscr{P}_{m-1}(\mathbb{F})$, Corollary 2.18 implies that $\mathscr{B}^{v_\Phi}$ is maximal among all $\mathscr{B}^w$ with w similar to v_Φ.

More precisely, consider the valuation ring

$$S := \{s\in\mathbb{F} \mid \Phi(s)\leqslant 1\},$$

its maximal ideal

$$\mathbf{m} := \{s\in S \mid \Phi(s)<1\},$$

and the residue class field $\mathbb{F}_\Phi := S/\mathbf{m}$. Then we have a *projective homomorphism* $\pi\colon \mathscr{P}_{m-1}(\mathbb{F})\to\mathscr{P}_{m-1}(\mathbb{F}_\Phi)$, defined by Klingenberg in [K] and given by

$$\pi\left(\mathbb{F}\cdot\begin{pmatrix} s_1\\ \vdots\\ s_m\end{pmatrix}\right) := \mathbb{F}_\Phi\cdot\begin{pmatrix} \bar{s}_1\\ \vdots\\ \bar{s}_m\end{pmatrix}$$

with $s_i\in S$ for all $i=1, ..., m$, $s_i\in S\setminus\mathbf{m}$ for at least one such i and $\bar{s}_i := s_i+\mathbf{m}$. For $\{e_1, ..., e_m\}\subseteq\mathscr{P}_{m-1}(\mathbb{F})$ we have $\{e_1, ..., e_m\}\in\mathscr{B}^{v_\Phi}$ if and only if $\{\pi(e_1), ..., \pi(e_m)\}$ is a base of the projective space $\mathscr{P}_{m-1}(\mathbb{F}_\Phi)$.

5. The Tutte Group of Valuated Matroids

In this section we want to show that some matroids are rigid. To this end we will have to make use of the Tutte group of a matroid which has been introduced in [DW1]. Moreover, we will have to recall some results about projective equivalence of matroids with coefficients stated in [W2, Sect. 6] in terms of the Tutte group. Most of the results proved in this section will be simple consequences of the basic techniques, developed in [DW1, W1, W2].

We repeat the "axiomatic" definition of the Tutte group as it has been given in [DW2, Sect. 3]: We assume that M is a combinatorial geometry of finite rank m, defined on a possibly infinite set E. Let $\mathscr{B}=\mathscr{B}_M$ denote the set of bases, $\mathscr{H}=\mathscr{H}_M$ the set of hyperplanes, $\mathscr{C}=\mathscr{C}_M$ the set of circuits, $\rho=\rho_M$ the rank function, and $\langle\cdots\rangle=\langle\cdots\rangle_M$ the closure operator of M. Furthermore, we put

$$\mathscr{B}_{(M)} := \{(a_1, ..., a_m)\in E^m \mid \{a_1, ..., a_m\}\in\mathscr{B}\},$$
$$\mathscr{H}_{(M)} := \{(H; a, b) \mid H\in\mathscr{H};\ a, b\in E\setminus H\},$$
$$\mathscr{C}_{(M)} := \{(C; a, b) \mid C\in\mathscr{C};\ a, b\in C\}.$$

THEOREM 5.1. *There exists an abelian group* $\mathbb{T} = \mathbb{T}_M$, *together with a specified element* $\varepsilon_M \in \mathbb{T}_M$ *and maps*

$$\text{(A)} \quad T_1 : \mathscr{B}^2_{(M)} \to \mathbb{T}_M,$$

$$\text{(B)} \quad T_2 : \mathscr{H}_{(M)} \to \mathbb{T}_M,$$

$$\text{(C)} \quad T_3 : \mathscr{C}_{(M)} \to \mathbb{T}_M$$

such that the following holds:

(i) $\varepsilon_M^2 = 1$;

(ii) $T_1((\mathbf{a}, \mathbf{b})) \cdot T_1((\mathbf{b}, \mathbf{c})) \cdot T_1((\mathbf{c}, \mathbf{a})) = 1$ *for all* $(\mathbf{a}, \mathbf{b})$, $(\mathbf{b}, \mathbf{c})$, $(\mathbf{c}, \mathbf{a}) \in \mathscr{B}^2_{(M)}$,

$$T_1((a_1, ..., a_m), (a_{\tau(1)}, ..., a_{\tau(m)})) = \begin{cases} 1 & \text{if } \tau \text{ is an even permutation in } \Sigma_m \\ \varepsilon_M & \text{if } \tau \text{ is an odd permutation in } \Sigma_m, \end{cases}$$

$$T_1((a_1, ..., a_{m-2}, b_1, c_1), (a_1, ..., a_{m-2}, b_1, c_2)) = T_1((a_1, ..., a_{m-2}, b_2, c_1), (a_1, ..., a_{m-2}, b_2, c_2))$$

if $\{a_1, ..., a_{m-2}, b_i, c_j\} \in \mathscr{B}$ *for* $i, j \in \{1, 2\}$, *but* $\{a_1, ..., a_{m-2}, b_1, b_2\} \notin \mathscr{B}$;

(iii) $T_2(H; a_1, a_1) = T_2(H; a_1, a_2) \cdot T_2(H; a_2, a_3) \cdot T_2(H; a_3, a_1) = 1$ *for* $H \in \mathscr{H}$, $a_1, a_2, a_3 \in E \backslash H$,

$$\varepsilon_M \cdot T_2(H_1; a_2, a_3) \cdot T_2(H_2; a_3, a_1) \cdot T_2(H_3; a_1, a_2) = 1$$

for $H_1, H_2, H_3 \in \mathscr{H}$, $L := H_1 \cap H_2 \cap H_3 = H_i \cap H_j$ *for* $i \neq j$, $\rho(L) = m - 2$ *and* $a_i \in H_i \backslash L$ *for* $i \in \{1, 2, 3\}$;

(iv) $T_3(C; a_1, a_1) = T_3(C; a_1, a_2) \cdot T_3(C; a_2, a_3) \cdot T_3(C; a_3, a_1) = 1$ *for* $C \in \mathscr{C}$, $a_1, a_2, a_3 \in C$,

$$\varepsilon_M \cdot T_3(C_1; a_2, a_3) \cdot T_3(C_2; a_3, a_1) \cdot T_3(C_3; a_1, a_2) = 1$$

for $C_1, C_2, C_3 \in \mathscr{C}$, $D := C_1 \cup C_2 \cup C_3 = C_i \cup C_j$ *for* $i \neq j$, $\rho(D) = \#(D) - 2$ *and* $a_i \in D \backslash C_i$ *for* $i \in \{1, 2, 3\}$;

(v) *If* $((a_1, ..., a_{m-1}, a), (a_1, ..., a_{m-1}, b)) \in \mathscr{B}^2_{(M)}$ *with* $a \neq b$, $C = \{x \in \{a_1, ..., a_{m-1}, a, b\} \mid \{a_1, ..., a_{m-1}, a, b\} \backslash \{x\} \in \mathscr{B}\}$ *and* $H = \langle \{a_1, ..., a_{m-1}\} \rangle$, *then*

$$T_1((a_1, ..., a_{m-1}, a), (a_1, ..., a_{m-1}, b)) = T_2(H; a, b) = \varepsilon_M \cdot T_3(C; b, a).$$

In particular, if $C \in \mathscr{C}$, $H \in \mathscr{H}$ *and* $C \backslash H = \{a, b\}$, *then*

$$T_2(H; a, b) = \varepsilon_M \cdot T_3(C; b, a).$$

(vi) *If* $\mathbb{T}'$ *is also an abelian group with* $\varepsilon' \in \mathbb{T}'$, $\varepsilon'^2 = 1$ *and* $T'_1: \mathscr{B}^2_{(M)} \to \mathbb{T}'$ (*or* $T'_2: \mathscr{H}_{(M)} \to \mathbb{T}'$ *or* $T'_3: \mathscr{C}_{(M)} \to \mathbb{T}'$) *satisfies* (ii) (*or* (iii) *or* (iv)), *then there exists a unique homomorphism* $\psi: \mathbb{T}_M \to \mathbb{T}'$ *with* $\psi(\varepsilon_M) = \varepsilon'$ *and* $T'_1 = \psi \circ T_1$ (*or* $T'_2 = \psi \circ T_2$ *or* $T'_3 = \psi \circ T_3$, *respectively*).

Proof. This is [DW2, Theorem 3.1] and summarizes the results of [DW1, Sect. 1]. ∎

Definition 5.2. The *Tutte group* of the combinatorial geometry M is the group $\mathbb{T}_M$ which by Theorem 5.1 is determined uniquely up to isomorphism.

Remark. The notations, introduced above, are related to the notations, introduced in [DW1, Sect. 1], in the following way:

$$\begin{aligned}
&T_1((a_1, ..., a_{m-1}, a), (a_1, ..., a_{m-1}, b)) \\
&\quad = T_{(a_1, ..., a_{m-1}, a)} \cdot T^{-1}_{(a_1, ..., a_{m-1}, b)} \\
&\qquad \text{for} \quad \{a_1, ..., a_{m-1}, a\}, \{a_1, ..., a_{m-1}, b\} \in \mathscr{B}; \\
&T_2(H; a, b) = T_{H,a} \cdot T^{-1}_{H,b} \qquad \text{for} \quad H \in \mathscr{H},\ a, b \in E \backslash H; \\
&T_3(C; a, b) = T_{C,a} \cdot T^{-1}_{C,b} \qquad \text{for} \quad C \in \mathscr{C},\ a, b \in C.
\end{aligned}$$

We have to consider a particular subgroup of $\mathbb{T}_M$. For $e \in E$ let $\delta_e: E \to \mathbb{Z}$ denote the map given by

$$\delta_e(f) := \begin{cases} 1 & \text{for } f = e \\ 0 & \text{for } f \neq e. \end{cases}$$

By [DW1, Theorem 1.2] it is immediate that $\mathbb{T}_M$ is generated by ε_M and all $T_2(H; a, b)$ for $H \in \mathscr{H}$, $a, b \in E \backslash H$.

Proposition and Definition 5.3. *For* $\mathbb{T}' := \mathbb{Z}^E$ *and* $\varepsilon' := 0$ *the map* $T'_2: \mathscr{H}_{(M)} \to \mathbb{T}': (H; a, b) \mapsto \delta_a - \delta_b$ *satisfies condition* (iii) *in Theorem* 5.1, *and therefore it induces a unique homomorphism* $\theta: \mathbb{T}_M \to \mathbb{Z}^E$, *satisfying* $\theta(\varepsilon_M) = 0$ *and* $\theta(T_2(H; a, b)) = \delta_a - \delta_b$ *for* $H \in \mathscr{H}$, $a, b \in E \backslash H$.

The kernel $\mathbb{T}^{(0)}_M := \ker \theta$ *of* θ *is called the* inner Tutte group *of* M.

Proof. All claims are perfectly trivial. ∎

For a matroid M with coefficients we put

$$\mathscr{B}_M := \mathscr{B}_{\underline{M}}, \qquad \mathscr{H}_M := \mathscr{H}_{\underline{M}}, \qquad \mathscr{C}_M := \mathscr{C}_{\underline{M}}, \qquad \mathbb{T}_M := \mathbb{T}_{\underline{M}},$$

where, as above, $\underline{M}$ denotes the underlying combinatorial geometry of M.

The next result relates a matroid M which is defined in terms of some Grassmann–Plücker map b with its Tutte group $\mathbb{T}_M$.

PROPOSITION 5.4. *Assume $M = M_b$ is a matroid of rank $m < \infty$ with coefficients in a fuzzy ring $K = (K; +; \cdot; \varepsilon; K_0)$ for some Grassmann–Plücker map $b\colon E^m \to K^* \cup \{0\}$. Then there exists a unique homomorphism $\varphi_b\colon \mathbb{T}_M \to K^*$ satisfying*

$$\varphi_b(\varepsilon_M) = \varepsilon, \tag{5.1a}$$

$$\varphi_b(T_1((e_1, ..., e_m), (f_1, ..., f_m))) = b(e_1, ..., e_m) \cdot b(f_1, ..., f_m)^{-1}$$
$$\text{for all} \quad (e_1, ..., e_m), (f_1, ..., f_m) \in \mathscr{B}_{(M)}. \tag{5.1b}$$

Proof. As above, this is a direct consequence of Theorem 5.1 (vi) or, as well, a trivial reformulation of [W2, Proposition 4.4]. ∎

In case $K = K_\Gamma$ we will once more make use of [DW6] to prove more precisely

THEOREM 5.5. *Assume that $K = K_\Gamma$ for some linearly ordered abelian group Γ and that M is a combinatorial geometry, defined on E. Then a homomorphism $\varphi\colon \mathbb{T}_M \to K^*$ with $\varphi(\varepsilon_M) = 1$ satisfies $\varphi = \varphi_b$ for some Grassmann–Plücker map $b\colon E^m \to K_\Gamma^* \cup \{0\}$ with $\underline{M_b} = M$ if and only if the following condition holds:*

If $e_1, ..., e_{m-2}, a_1, ..., a_4 \in E$ are such that $\{e_1, ..., e_{m-2}, a_i, a_j\}$ is a base of M for all $1 \leqslant i < j \leqslant 4$, then for

$$s_1 := \varphi(T_1((e_1, ..., e_{m-2}, a_1, a_2), (e_1, ..., e_{m-2}, a_1, a_3)))$$
$$\cdot \varphi(T_1((e_1, ..., e_{m-2}, a_4, a_3), (e_1, ..., e_{m-2}, a_4, a_2)))$$

and

$$s_2 := \varphi(T_1((e_1, ..., e_{m-2}, a_3, a_2), (e_1, ..., e_{m-2}, a_3, a_1)))$$
$$\cdot \varphi(T_1((e_1, ..., e_{m-2}, a_4, a_1), (e_1, ..., e_{m-2}, a_4, a_2)))$$

we have

$$1 \leqslant \max(s_1, s_2)$$

and

$$1 = \max(s_1, s_2) \qquad \textit{if} \quad s_1 \neq s_2.$$

Proof. The theory of matroids with coefficients in a perfect fuzzy ring as developed in [DW6] (cf. in particular [DW6, Theorem 3.8]) shows that a homomorphism $\varphi\colon \mathbb{T}_M \to K^*$ with $\varphi(\varepsilon_M) = \varepsilon = 1$ satisfies $\varphi = \varphi_b$ for some Grassmann–Plücker map $b\colon E^m \to K^* \cup \{0\}$ if and only if

$$s_1 + s_2 + \varepsilon = s_1 + s_2 + 1 \in K_0$$

for any possible choices of $s_1, s_2 \in K^*$ as above. Thus our theorem follows from Lemma 4.6. ∎

We will now see that projective equivalence of valuations of matroids as considered in Section 2 fits perfectly well into the more general concept of projective equivalence of matroids with coefficients as considered in [W2].

Definition 5.6. Assume $K = (K; +; \cdot; \varepsilon; K_0)$ is some fuzzy ring. Two Grassmann–Plücker maps $b_1, b_2\colon E^m \to K^* \cup \{0\}$ or the matroids M_{b_1} and M_{b_2} are called *projectively equivalent*, if there exists some $\alpha \in K^*$ and some map $\eta\colon E \to K^*$ with

$$b_1(e_1, ..., e_m) = \alpha \cdot \prod_{i=1}^{m} \eta(e_i) \cdot b_2(e_1, ..., e_m) \qquad \text{for all} \quad e_1, ..., e_m \in E. \tag{5.2}$$

Remarks. (i) If b_1 and b_2 are projectively equivalent, then it is clear that b_1 and b_2 define the same combinatorial geometry, because for $(e_1, ..., e_m) \in E^m$ we have $b_1(e_1, ..., e_m) \neq 0$ if and only if $b_2(e_1, ..., e_m) \neq 0$.

(ii) By [W2, Proposition 5.1] projective equivalence as defined in Definition 5.6 means the same as defined in [W2], if the underlying combinatorial geometry has finite rank m.

(iii) If $K = K_\Gamma$ for some linearly ordered abelian group Γ, then by Theorem 4.3 projective equivalence of valuations means the same as projective equivalence of Grassmann–Plücker maps.

In the sequel we assume that M is some fixed combinatorial geometry, defined on E and of rank m. If $K = (K; +; \cdot; \varepsilon; K_0)$ is a fuzzy ring, then a Grassmann–Plücker map $b\colon E^m \to K^* \cup \{0\}$ is called a Grassmann–Plücker map for M, if $\mathscr{B}_M$ is the set of bases of M_b; that means $\underline{M_b} = M$.

We have the following basic

Theorem 5.7. *Assume $K = (K; +; \cdot; \varepsilon; K_0)$ is a fuzzy ring. Two Grassmann–Plücker maps $b_1, b_2\colon E^m \to K^* \cup \{0\}$ for the combinatorial geometry M are projectively equivalent if and only if the homomorphisms*

$\varphi_{b_1}, \varphi_{b_2}: \mathbb{T}_M \to K^*$ given by Proposition 5.4 coincide on the inner Tutte group $\mathbb{T}_M^{(0)}$.

Proof. This is a simple consequence of our definitions, see, for instance, [W2, Theorem 6.3]. ∎

Theorem 5.7 implies

COROLLARY 5.8. *Assume $K = (K; +; \cdot; \varepsilon; K_0)$ is a fuzzy ring such that there exists at most one homomorphism $\varphi: \mathbb{T}_M^{(0)} \to K^*$ with $\varphi(\varepsilon_M) = \varepsilon$. (This assumption holds if K^* is torsion free and $\mathrm{T}_M^{(0)}$ is a torsion group.) Then there exists at most one projective equivalence class of Grassmann–Plücker maps $b: E^m \to K^* \cup \{0\}$ with M as its underlying combinatorial geometry.*

We want to show that binary matroids as well as finite projective spaces are rigid. To this end we recall the following two results.

PROPOSITION 5.9 (see [W1, Theorem 5.2]). *Assume the combinatorial geometry M is binary. Then we have*

$$\mathbb{T}_M^{(0)} = \langle \varepsilon_M \rangle \cong \begin{cases} \{0\} & \textit{if the Fano-Matroid or its dual is a minor of } M \\ \mathbb{Z}/2\mathbb{Z} & \textit{otherwise.} \end{cases}$$

PROPOSITION 5.10 (see [DW5, Corollary 3.8]). *Assume $\mathscr{P}$ is a finite projective space of dimension at least 2 and M is its underlying combinatorial geometry. Then $\mathbb{T}_M^{(0)}$ is finite.*

Corollary 5.8, Proposition 5.9, and Proposition 5.10 imply

THEOREM 5.11. (i) *If $\mathbb{T}_M^{(0)}$ has no elements of infinite order, then M is rigid. In particular, M is rigid if $\mathbb{T}_M^{(0)}$ is finite.*

(ii) *Every binary combinatorial geometry is rigid.*

(iii) *Every finite projective space of dimension at least two is rigid.*

Remark. In Example 2.5 we have seen that the uniform matroid $U_{2,4}$ is not rigid. However, every proper minor of $U_{2,4}$ is binary, and thus it is rigid. Therefore $U_{2,4}$ is the unique minimal matroid which is not rigid, because a matroid M is non-binary if and only if M contains $U_{2,4}$ as a minor. However, note that Theorem 5.11(iii) together with the fact that $U_{2,4}$ is a minor of any projective plane of order at least three shows that a combinatorial geometry may be rigid although this does not hold for every minor.

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